Task sheet 1

Task 1. Show that a satisfiability of a system of word equations is NP-hard already when $\Sigma = \{a\}$.

Hint: This reduces to some other known equations.

Task 2. Show that the satisfiability of word equations is NP-hard when we consider only systems in which every right-hand side does not contain variables.

(Note: it might be easier to show this when we allow also ϵ as a substitution for a variable).

Task 3. Show that the problem of satisfiability of a system of word equations can be reduced to the problem of satisfiability of a single word equation, when we are allowed to add letters to the alphabet. Show the same result also when adding letters is not allowed, but $|\Sigma| \geq 2$.

Task 4. Suppose that s is a length-minimal solution of a word equation $u = v$. Let w be a substring of $s(u)$. Show that w has an occurrence that touches a cut.

Strengthen this for $|w| \geq 2$: in this case w overlaps a cut.

Strengthen this for $w = a \in \Sigma$: in this case a occurs in u or in v.

Conclude that without loss of generality the length-minimal solutions do not use letters outside the alphabet Σ.

Hint: Using the inductive definition of the transitive closure may be helpful.

Task 5. Reduce the satisfiability problem for word equations to the satisfiability problem of cubic word equations, i.e. when each variable occurrs at most three times in the system of word equations.

Task 6. Show that the problem of word equations with context-free constraints is undecidable. Here "context-free constraints" means that we apart from the equations, we allow also condition of the form $X \in L$, where X is any variable and L is a CFG. For s to be a solution we require then that $s(X) \in L$.

Task 7. Show that the Intersection Problem for DFAs

Problem: Non-emptiness of Intersection for DFAs
Input: DFAs (deterministic finite automata) $D_1, D_2, \ldots, D_m$
Task: Decide, whether the intersection of their languages is non-empty

is PSPACE-hard.
Show that this problem is in PSPACE even when we allow NFAs.
Deduce from this that word equations with regular constraints are PSPACE-hard; as in Task 6 the regular constraints mean that apart from the equations, we allow also condition of the form $X \in L$, where X is any variable and L is regular language, given a by a DFA and we require that $s(X) \in L$.

Task 8 ((Long: two points)). Consider a mapping from $\Sigma = \{a, b\}$ to 2×2 matrices over $\mathbb{N}$, defined as

$$\varphi(a) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \varphi(b) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad .$$

Extend this to Σ^* as a homomorphism.

Show that for any $w \in \Sigma^*$ its image is a a matrix with a determinant one.

Show that this mapping is injective; to do this, consider, what are the rows of a matrix $\varphi(a) \begin{bmatrix} n & n' \\ m & m' \end{bmatrix}$ and what are the rows of $\varphi(a) \begin{bmatrix} n & n' \\ m & m' \end{bmatrix}$. Deduce from this that looking at the matrix $M_w = \varphi(w)$ we can determine the left-most letter of w by looking at rows of M_w.

Show that if a 2×2 matrix M with determinant 1 and all natural entries can be represented as either $\varphi(a)M'$ or $\varphi(b)M'$, where M' has a determinant 1 and all natural entries. Again: compare the rows.

Deduce from this that φ is an isomorphism between Σ^* and 2×2 matrices with determinant 1 and all entries natural.

Deduce from this that satisfiability of word equation over $\Sigma = \{a, b\}$ reduces to the satisfiability of equations over natural numbers (to do this, represent a $\varphi(X)$ as a matrix of variables representing natural numbers).

Task 9. Show that the composition system of size n can be turned into an SLP of polynomial size. How small you can make the polynomial?