Task sheet 3

Task 20 (Newman's lemma). A rewriting system $S = \{(\ell_i, r_i)\}_{i \in I}$ is confluent, if for all s, t, u with $s \to_S^* t$ and $s \to_S^* u$ there exists v with $t, u \to_S^* v$; it is local confluent, if s, t, u with $s \to_S t$ and $s \to_S u$ there exists v with $t, u \to_S^* v$.

S is terminating, if there is no infinite chain

$$s_1 \to s_2 \to \cdots \to s_n \to \cdots .$$

Show that if S is locally confluent and terminating then it is confluent.

Task 21. An *involution* $\bar{\cdot}$ is any operation (defined in a semigroup) such that $\bar{\bar{\cdot}}$ is an identity and $\overline{ab} = \bar{b}\,\bar{a}$. In particular, we can define on some letters as an identity, such letters are called self-involuting.

Show that we can reduce a problem of word equations in a free semigroup with involution and regular constraints to the case in which there is no self-involuting letter.

Here regular constraints are defined using a homomorphism from the free semigroup with involution to a finite semigroup (with involution).

Task 22. Show that if a homomorphism $\rho : M \to \mathbb{B}_{n\times n}$ (so: Boolean matrices of size $n \times n$) from a free monoid with involution M into Boolean matrices does not preserve involution (in particular, the involution on $\mathbb{B}_{n\times n}$ may be undefined), then we can find a different set of Boolean matrices $\mathbb{B}_{m\times m}$ for which the involution is defined and there is a homomorphism $\rho' : M \to \mathbb{B}_{m\times m}$ from M to $\mathbb{B}_{m\times m}$ that preserves the involution and for each set of the form $\rho^{-1}(M)$ for some $M \subseteq \mathbb{B}_{n\times n}$ there is $M' \subseteq \mathbb{B}_{m\times m}$ such that $\rho^{-1}(M) = {\rho'}^{-1}(M')$ (but not necessarily the other way around), i.e. regular sets defined using ρ can be also defined using ρ'.

Hint: Take $\overline{\mathbb{B}_{n\times n}}$ and consider $\mathbb{B}_{n\times n} \times \overline{\mathbb{B}_{n\times n}}$. How to define the involution?

Task 23 ((2 points)). Show that given a word equation over a free monoid with regular constraints given by ρ we can extend the input alphabet Σ by letters

$$\{a_\tau \;:\; \tau \in N \text{ and there is a word } w \in \Sigma^* \text{ such that } \rho(w) = \tau\}.$$

Show the equisatisfiability of the problem over the original alphabet and over such an extended alphabet. Modify the algorithm that tests the satisfiability of word equations so that it works also in case of regular constraints. Can you implement the algorithm in PSPACE?