Task sheet 6

Task 31. Show that Strong Periodicity Lemma (so when w has periods p, q such that $|w| - gcd(p, q) \geq p+q$) follows from its variant in which $gcd(|u|, |v|) = 1$.

Task 32. Prove Strong Periodicity Lemma, it may be easiest to prove the variant with uv and vu having long enough prefix by adapting the proof for the case of $uv = vu$.

Task 33 ((Alternative proof of Periodicity Lemma). Given a word $w[1 \mathinner{..} p+q]$ with periods p, q such that $\text{nwd}(p, q) = 1$ define a graph on the positions of this word: there is an edge $\{i, j\}$ if and only if $|i - j| \in \{p, q\}$. Show that this graph is a cycle. Deduce from this that $w \in a^*$ for appropriate a.

Strengthen this to the case, when $w = w[1 \mathinner{..} p+q-1]$.

Hint: What happens with the graph from the first point, when we remove the last node?

Task 34. SHow that when u, v, w are primitive such that u^2 is a prefix of v^2 and v^2 of w^2 then $|u|+|v| \leq |w|$.

Task 35. Show that given a word w there are $\mathcal{O}(\log |w|)$ different primitive p such that p^2 is a prefix of w and that all such p can be found in $\mathcal{O}(|w|)$ time.

Task 36. Recall the linear-time construction of the MP array.

Task 37. Show that the exponential bound on the exponent of periodicity is tight (the exact constant at the exponent is not tight, though).

Task 38. Show that we can use the P-presentation approach for the compression algorithm (for solving word equations): we do not guess the lengths of the a-prefixes and suffixes, but denote them as variables and we write an appropriate system of linear equations.

Show that when the word equation can be encoded using m bits (in a natural encoding) then the constructed system has size $\mathcal{O}(m)$ bits.

Hint: Unary encoding the constants, in which a constant p is encoded using p bits, may be easier for proof purposes, even though it is not efficient.

Task 39. Show that we can verify the system of linear Diophantine equations in which all constants are encoded in unary in linear space (counted in bits).

Hint: Repeatedly guess the parity of sides of all equations and divide by 2. This is a first step for decidability of Presburger's arithmetics.

Task 40. Using the bound on the size of the minimal solutions of integer programming show that the doubly exponential bound on the size of the length-minimal solution follows from the original algorithm for satisfying word equations.

Task 41. Show that the P-presentation of a word is unique.

Task 42. Show that (for a given P) the P-presentation of a word w can be computed in $\mathcal{O}(|w|)$ time.