Task sheet 7

Task 43. Consider an equation

$$A_0 X A_1 \dots A_{n_X-1} X A_{n_X} = X B_1 \dots B_{n_X-1} X$$

with $A_0 \neq \epsilon \neq A_{n_X}$. Show that it has an equivalent equation in which $A'_{n_X} = \epsilon$ and $B'_{n_X} \neq \epsilon$.

Hint: First show that there is a system of equivalent equations, obtained by appropriate splitting, and then concatenate them in a different way.

Task 44. Show that if a word equation $Xp = qX$ is satisfiable then:

- p, q are conjugate and consequently also the primitive roots of p, q are conjugate, that is, there are u, v such that uv, vu are primitive and $p = (vu)^k$ and $q = (uv)^k$ for some $k \geq 1$;
- the set of solutions is $(uv)^* u$.

Given p, q the u, v can be calculated in linear time.

Task 45. Show that for a given a set of primitive words $P_1, \dots, P_k$ such that for each i $P_i^2 \sqsubseteq B_1 A_0 A_0$, in total time $\mathcal{O}(|B_1 A_0 A_0|)$ we can establish for all P_i from $P_1, \dots, P_k$ the P_i-prefix of $B_1 A_0 A_0$.

Task 46. Show that if a word equation

$$A_0 X A_1 \dots A_{n_X-1} X = X B_1 \dots B_{n_X-1} X B_{n_X}$$

has an infinite family of solutions $s_i(X) = (uv)^i u$, then uv is the primitive root of A_0 (this more or less follows from the two first cases of our analysis) and, by symmetry, vu is the primitive root of B_{n_X}.

Moreover, this equation is equivalent to a system of equations (as above), in which each equation additionally satisfies:

$$A_0 \in (uv)^+$$
$$A_i, B_j \in v(uv)^* \text{for } i > 0, j < n_X$$
$$B_{n_X} \in (vu)^+$$

To be more precise, we can partition the above equation into such a system.

To this end consider the uv-prefixes of the sides. What happens, when they terminate?

Task 47. Show that if a word equation

$$A_0 X A_1 \dots A_{n_X-1} X = X B_1 \dots B_{n_X-1} X B_{n_X}$$

has an infinite family of solutions $s_i(X) = (uv)^i u$ (for each i) then it has no other solution.

Use Task 46 and Task 44 and think big:

- use simplification from Task 46
- under those assumptions show that $X A_1 X \dots A_{n_X-1} X, X B_1 \dots B_{n_X-1} X \in (uv)^* u$: treat the equation as an equation from Task 44
- conclude that without loss of generality $A_1, \dots, A_{n_X-1}, B_1, dots, B_{n_X-1} = v$
- then $A_1 X \dots A_{n_X-1} X$ has period uv and Xv
- conclude that $X \in (uv)^* u$
- show that

. A proof by case inspection and periodicity is most likely very difficult. Laine and Plandowski [?] has such a proof, but it seems to have an error.

Task 48. Suppose that given a word w we can construct in $\mathcal{O}(n)$ time a structure such that given two indices i, j in $\mathcal{O}(1)$ time returns the length of the longest common prefix of words $w[i \mathinner{..} n]$ and $w[j \mathinner{..} n]$.

Explain how it can be used to verify in $\mathcal{O}(n + n_X \log n)$ time the $\mathcal{O}(\log n)$ candidate solutions, each of which is a prefix of A_0.

Task 49. Show that for every solution s of a word equation such that $s(X) \neq \epsilon$ the first letter of $s(X)$ is the first letter of A_0 and the last the last letter of $B_{n_{\mathcal{X}}}$.

If $A_0 \in a^+$ then $s(X) \in a^+$ for each solution s of $\mathcal{A} = \mathcal{B}$.

If the first letter of A_0 is a and $A_0 \notin a^+$ then there is at most one solution $s(X) \in a^+$, existence of such a solution can be tested (and its length returned) in $\mathcal{O}(|\mathcal{A}| + |\mathcal{B}|)$ time. Furthermore, for $s(X) \notin a^+$ the lengths of the a-prefixes of $s(X)$ and A_0 are the same.