Task sheet 11

17.01.2025

Task 66 (-). Write a polynomial with integer coefficients (in several variables $x, x_1, \ldots$) such that for each natural $n \geq 0$ there is an integer solution $(n, n_1, \ldots)$ and for each solution $(n, n_1, \ldots)$ we have $n \geq 0$.
This is used to guarantee that the solutions in the proof of Ehrenfeucht's conjecture are indeed positive.

Hint: Lagrange's four-square theorem.

Task 67. ~~Extend the proof of Ehrenfeucht's conjecture so that it allows regular constraints. Note, care is needed depending on the formalisation of "regular constraints." The easiest is through matrices, but so-far we treated transition matrices as Boolean matrices and now we use integer ones; this is not a problem. though. Multiply the transition matrix with appropriate vector to verify, whether we end up in an accepting state.~~

Task 68. Show that Ehrenfeucht's conjecture holds for arbitrary alphabets (of size at least 2). ~~You can use the Task 67 and encode larger alphabets in smaller ones plus regular constraints. This can be done directly as well, without regular constraints, but this is more complex.~~

Task 69. Show that a semigroup $S \subseteq \Sigma^*$ is right-unitary (its base is a prefix code) if and only if

$$\forall u \in S, v \in \Sigma^+ : uv \in S \implies v \in S.$$

Do not use the algorithm for computing the base (as its uses this condition).

Task 70 (2 points). Show that a semigroup $S \subseteq \Sigma^*$ is free (its base is a code) if and only if

$$\forall u \in \Sigma^+ : Su \cap S \neq \emptyset \quad \text{and} \quad uS \cap S \neq \emptyset \implies u \in S.$$

Do not use the algorithm for computing the base (as its uses this condition).

Task 71. Show that a base $B(S)$ of a semigroup S is a code if and only if S isomorphic to Γ^* for some finite Γ.

Task 72. Give a polynomial-time algorithm for verifying, whether (finite) $A \subseteq \Sigma^*$ is a code.

Hint: Automata; fixing two words that should give a counterexample may help.

Task 73. Give a polynomial-time algorithm for verifying, whether (finite) $A \subseteq \Sigma^*$ is an ω-code.

Hint: Automata over infinite strings are one option; using some periodicity is another. Or you can use automata over finite words directly.

Task 74. Show that the base of smallest semigroup containing A, so $B(F(A))$, consists of suffixes of some words from A. Show that they are in fact borders of words from A.

Hint: Easy for suffixes. For borders: use a trick to show that they are prefixes and then the fact that the base is unique.