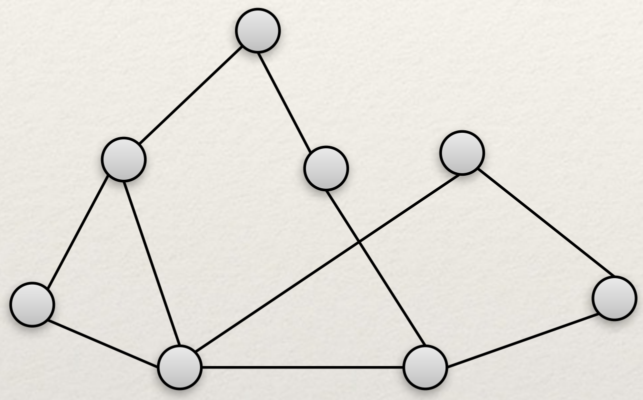
Approximating Graphs by Trees

Marcin Bieńkowski

The problem

Given an arbitrary graph
$G = (V,E)$ of n nodes…



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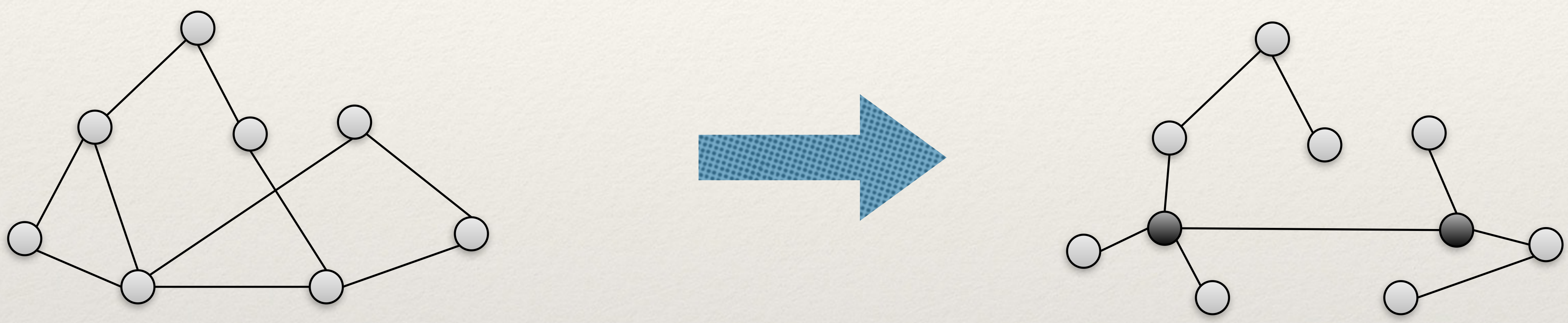

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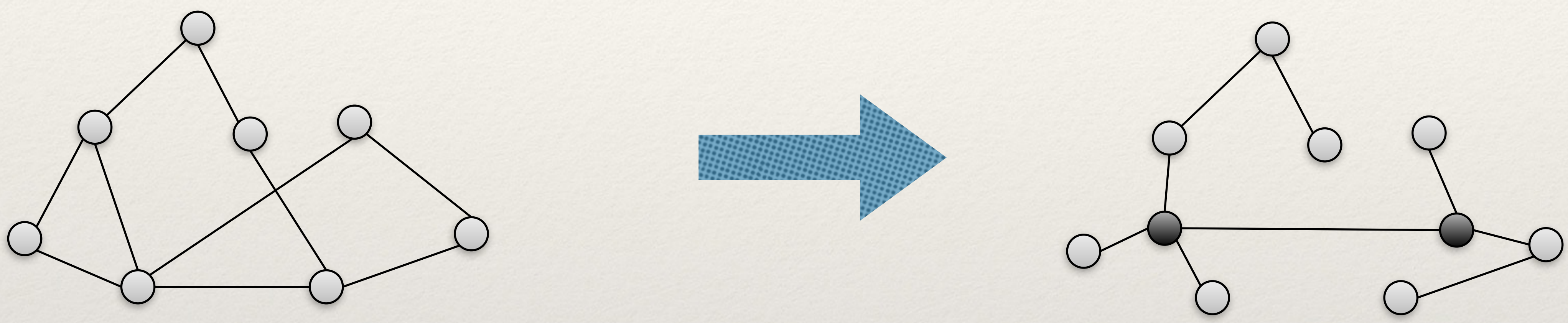

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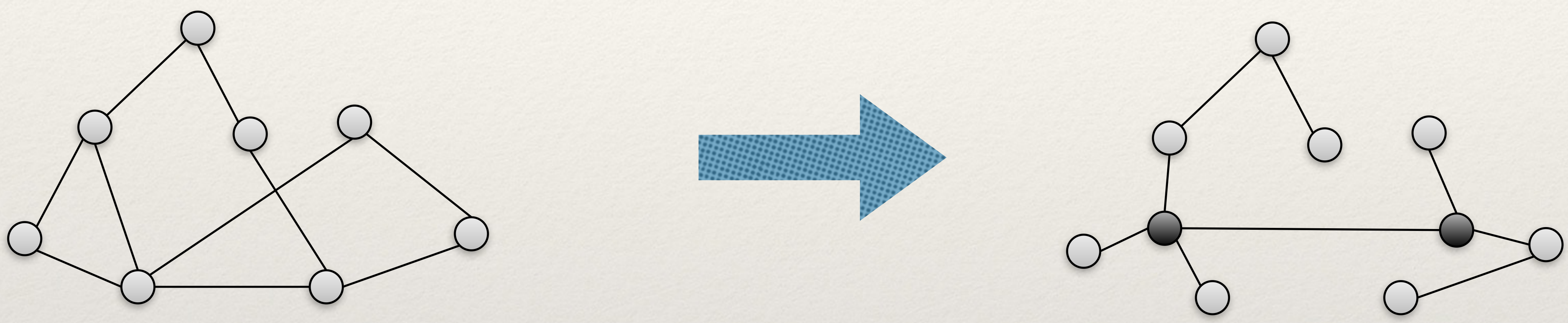

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distortion

Applications…

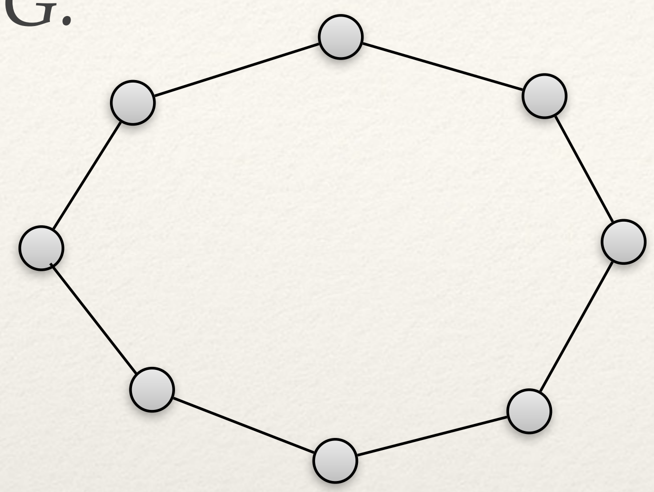
… for most of optimization problems on graphs, whose objective function is linear in distances.

In particular:

- metrical task systems
- group Steiner tree
- buy-at-bulk network design
- distributed paging
- k-server
- k-median
- …

Is small distortion possible? (1)

G:



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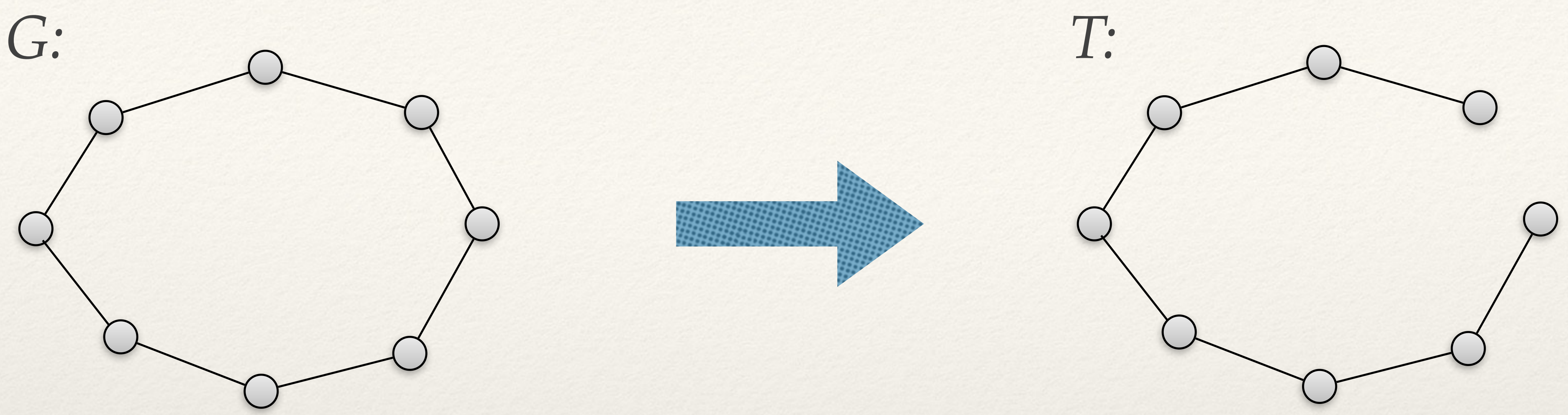

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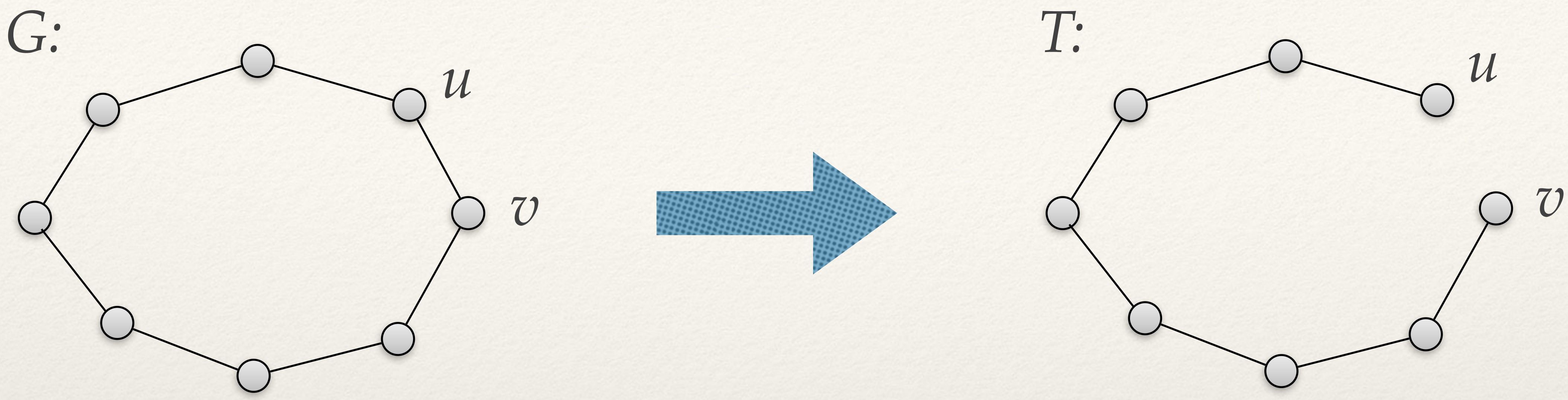

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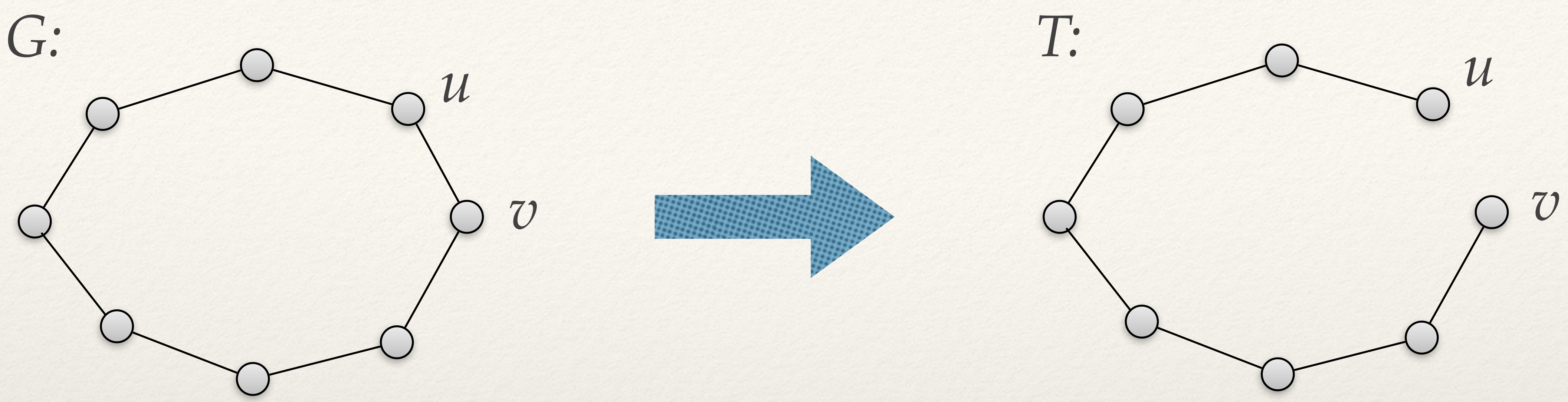

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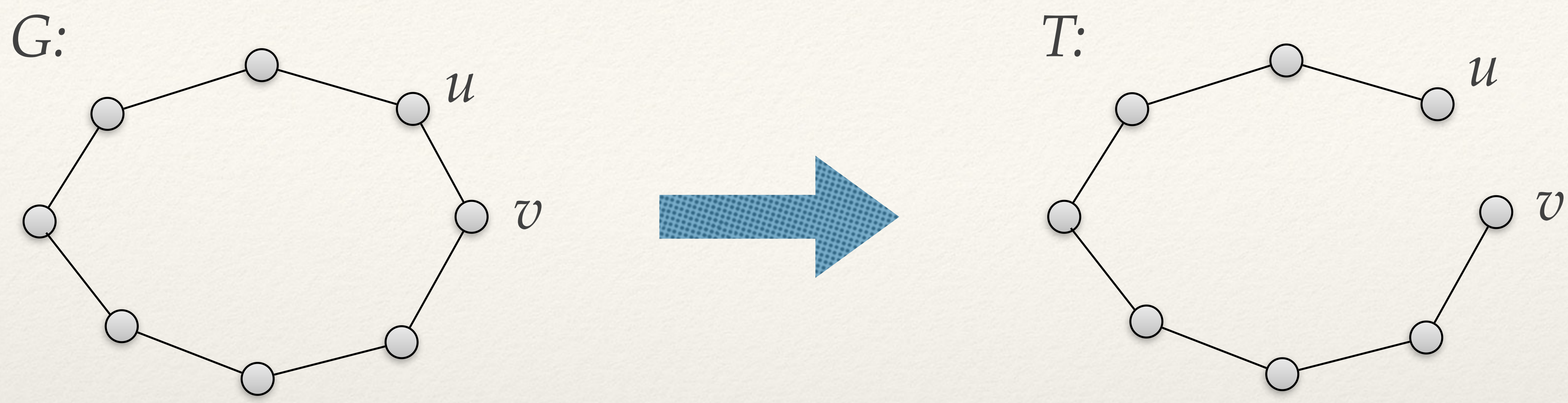

- The distortion is *n-1*.
- Actually, this is more complicated as *T* may contain additional nodes, but the distortion is Ω(n). **[Rabinovich, Raz 2005]**
- No good **deterministic** choice of *T*, what about **random** one?

Is small distortion possible? (2)

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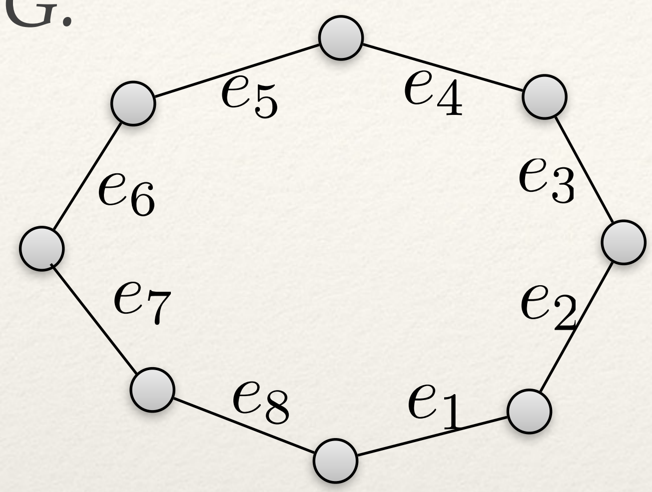

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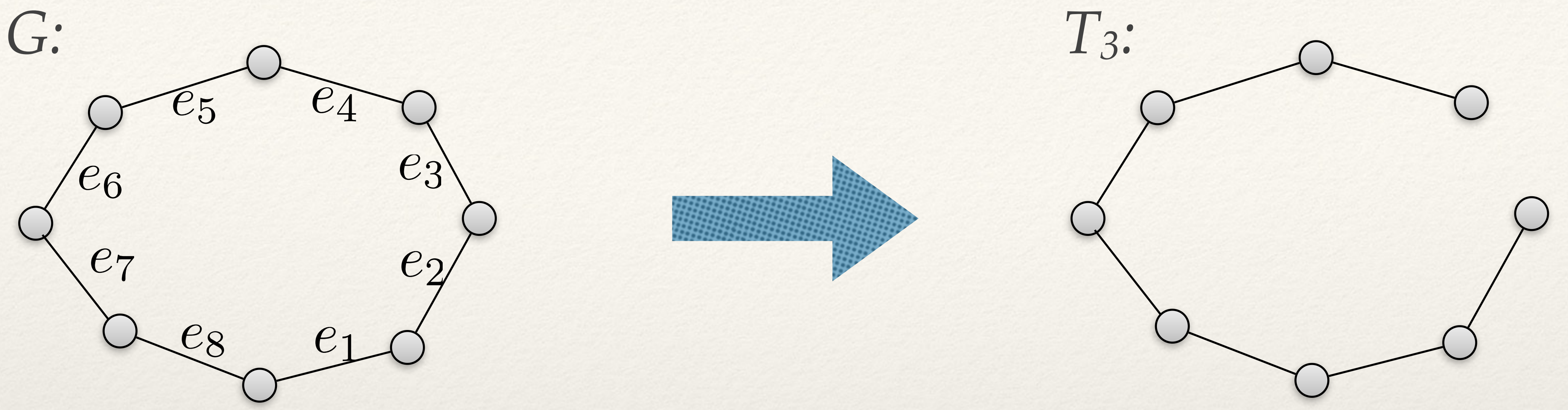

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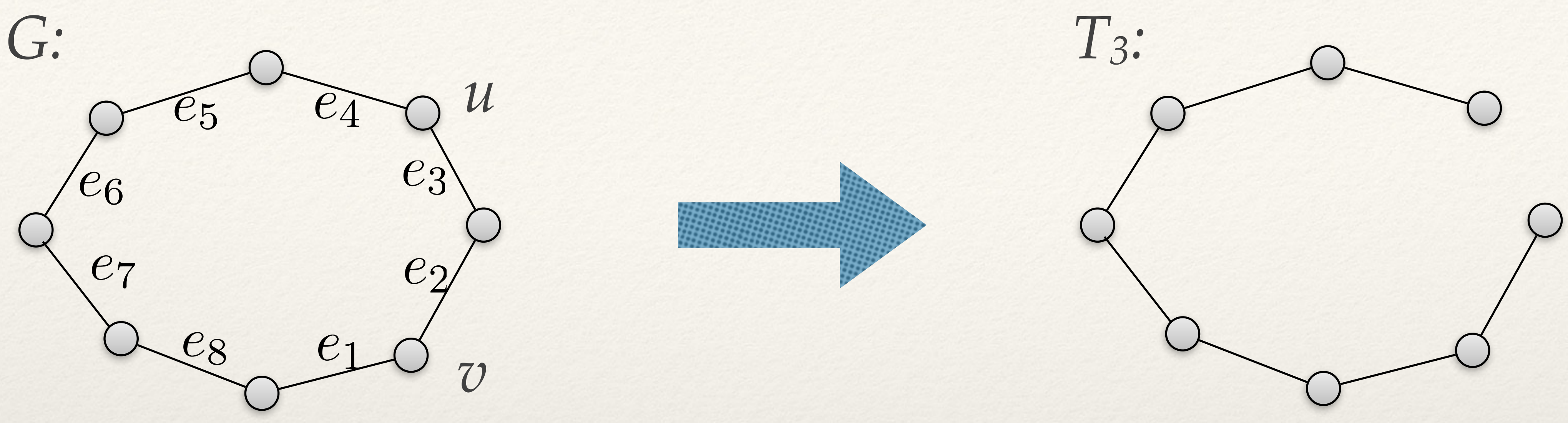

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- Fix any two nodes u and v, let $k = d_{u,v}$.

Then, $\mathbf{E}[T_{u,v}] = \frac{k}{n} \cdot (n-k) + \frac{n-k}{n} \cdot k = 2k \cdot \frac{n-k}{n} < 2k$

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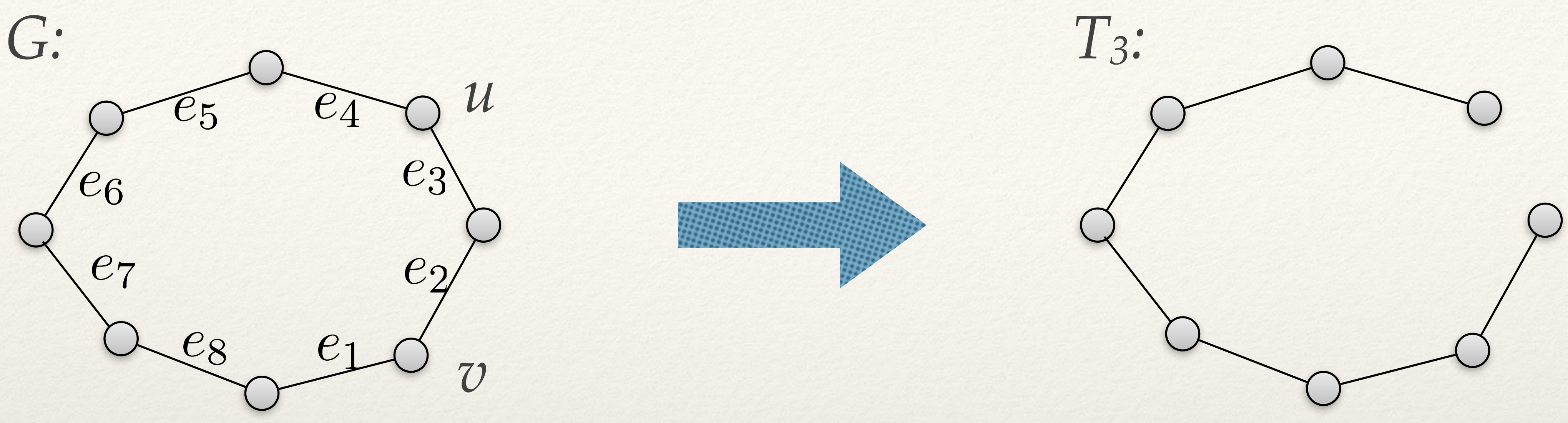

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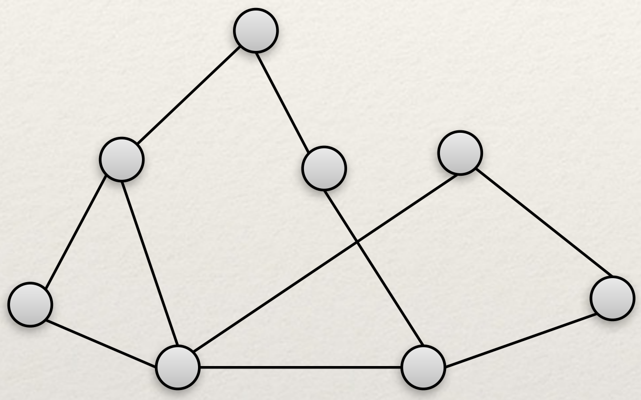

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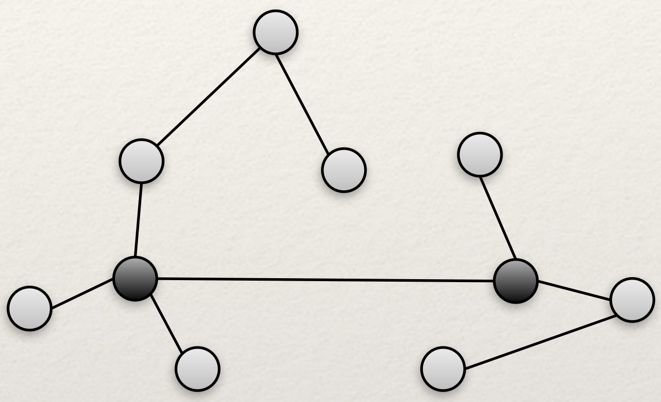

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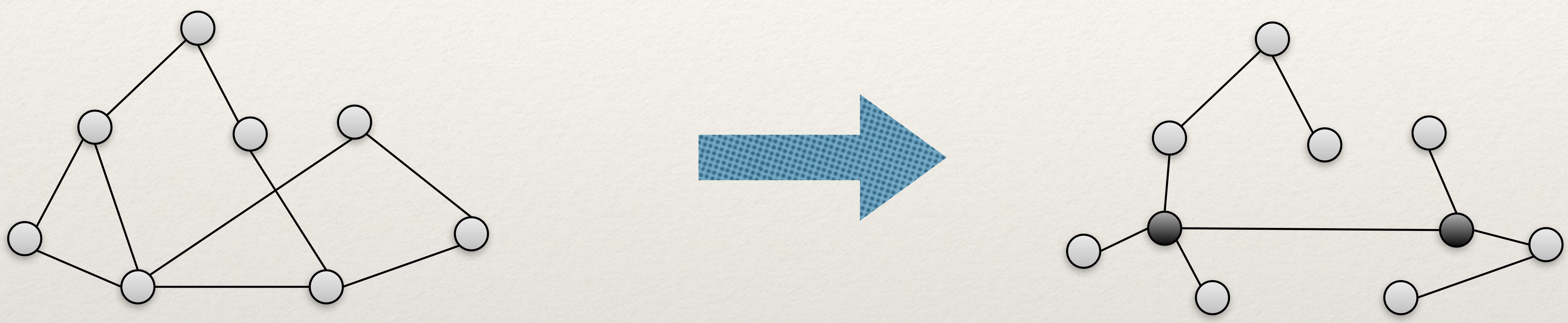

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Historical Notes

Bounding distortion:

- $2^{O(\sqrt{\log n \log\log n})}$ **[Alon, Karp, Peleg, West, SIAM Jcomp '91]**
- $O(\log^2 n)$ **[Bartal, FOCS '96]**
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asymptotically optimal

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Hierarchical decomposition

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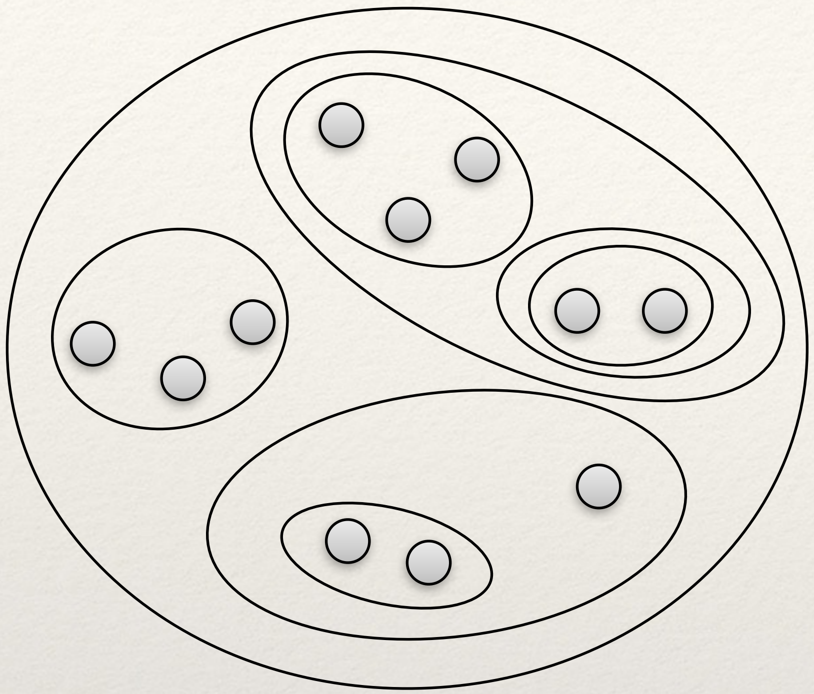
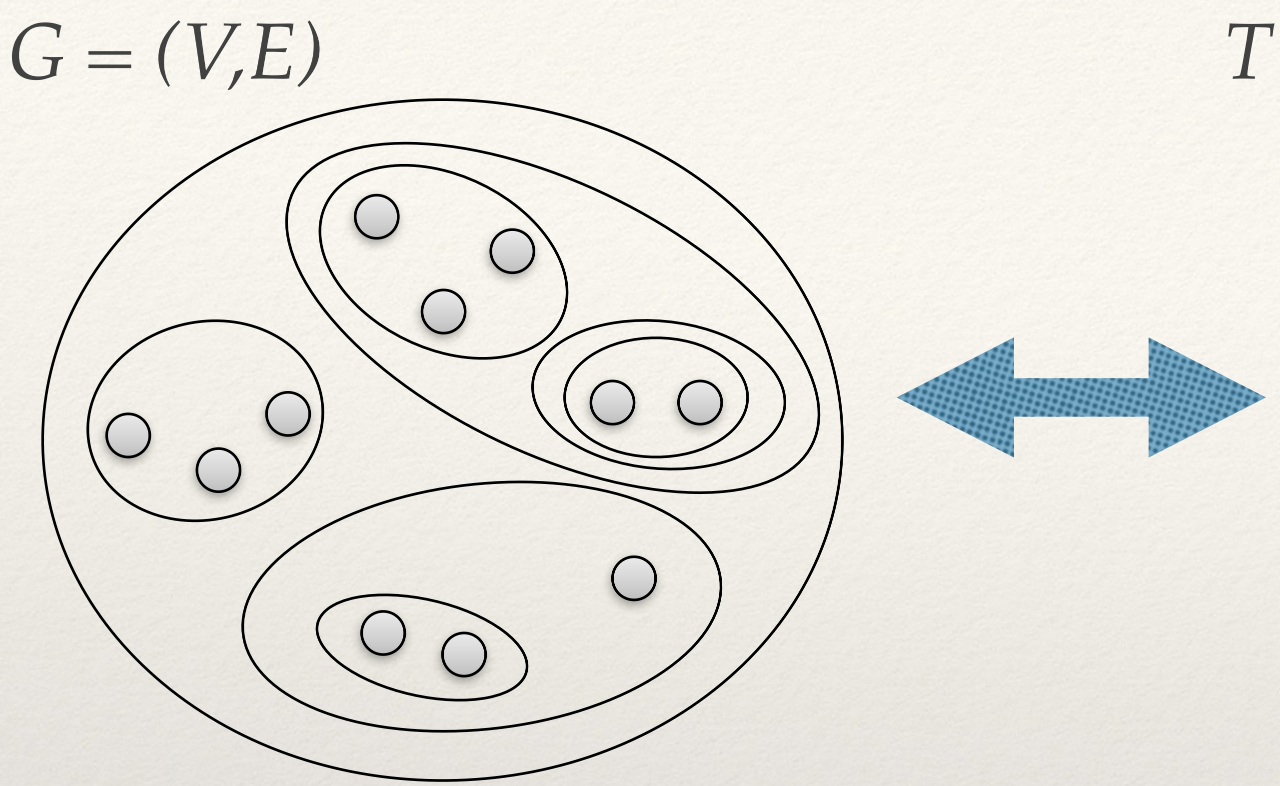
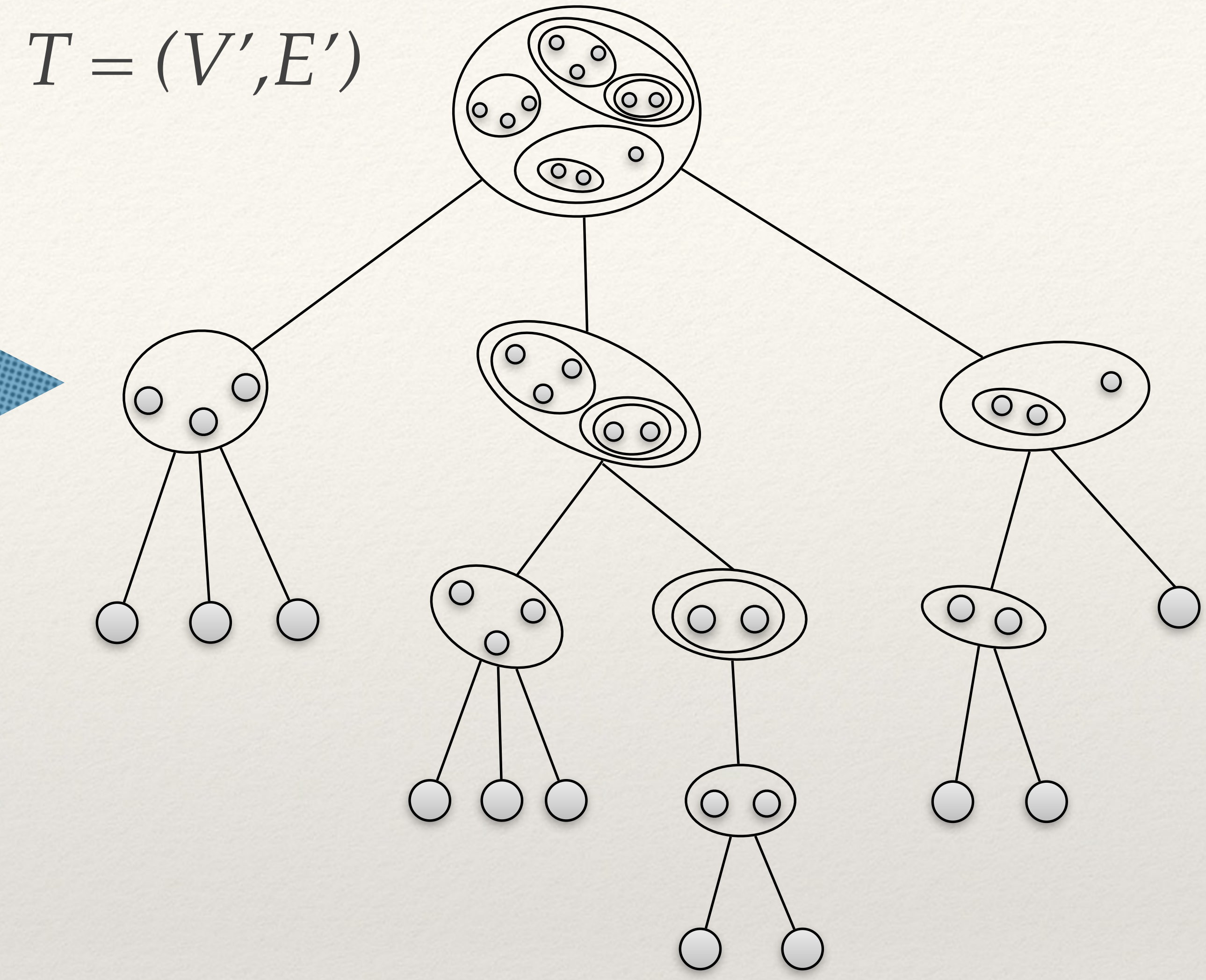


- Iteratively partition set V

Hierarchical decomposition





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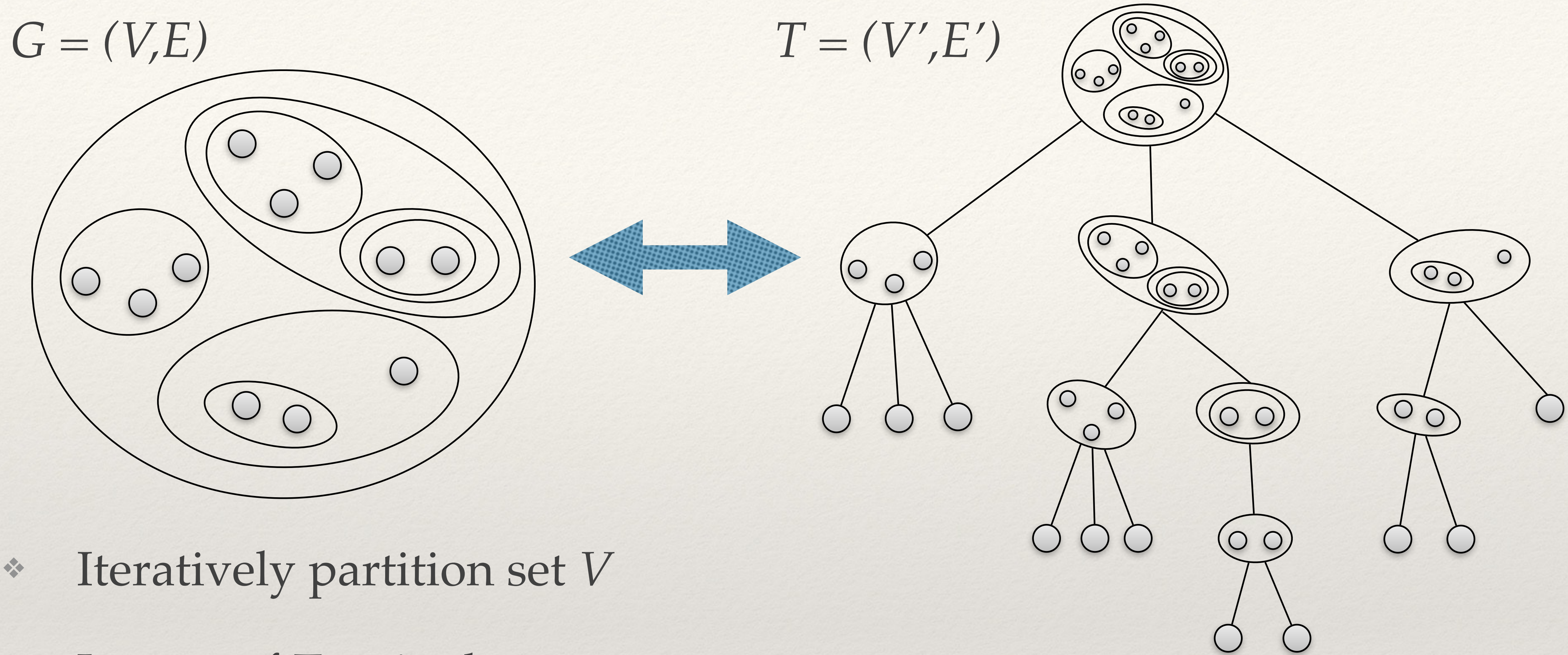

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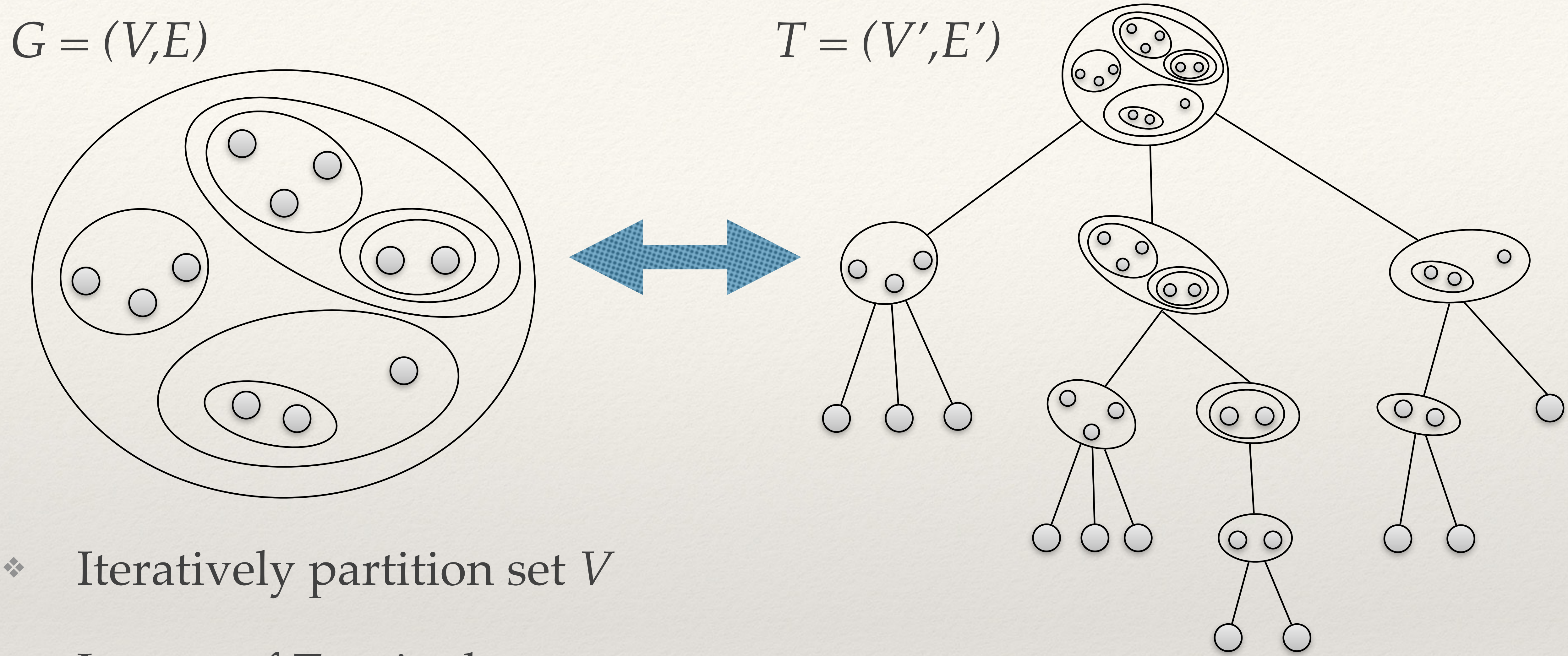

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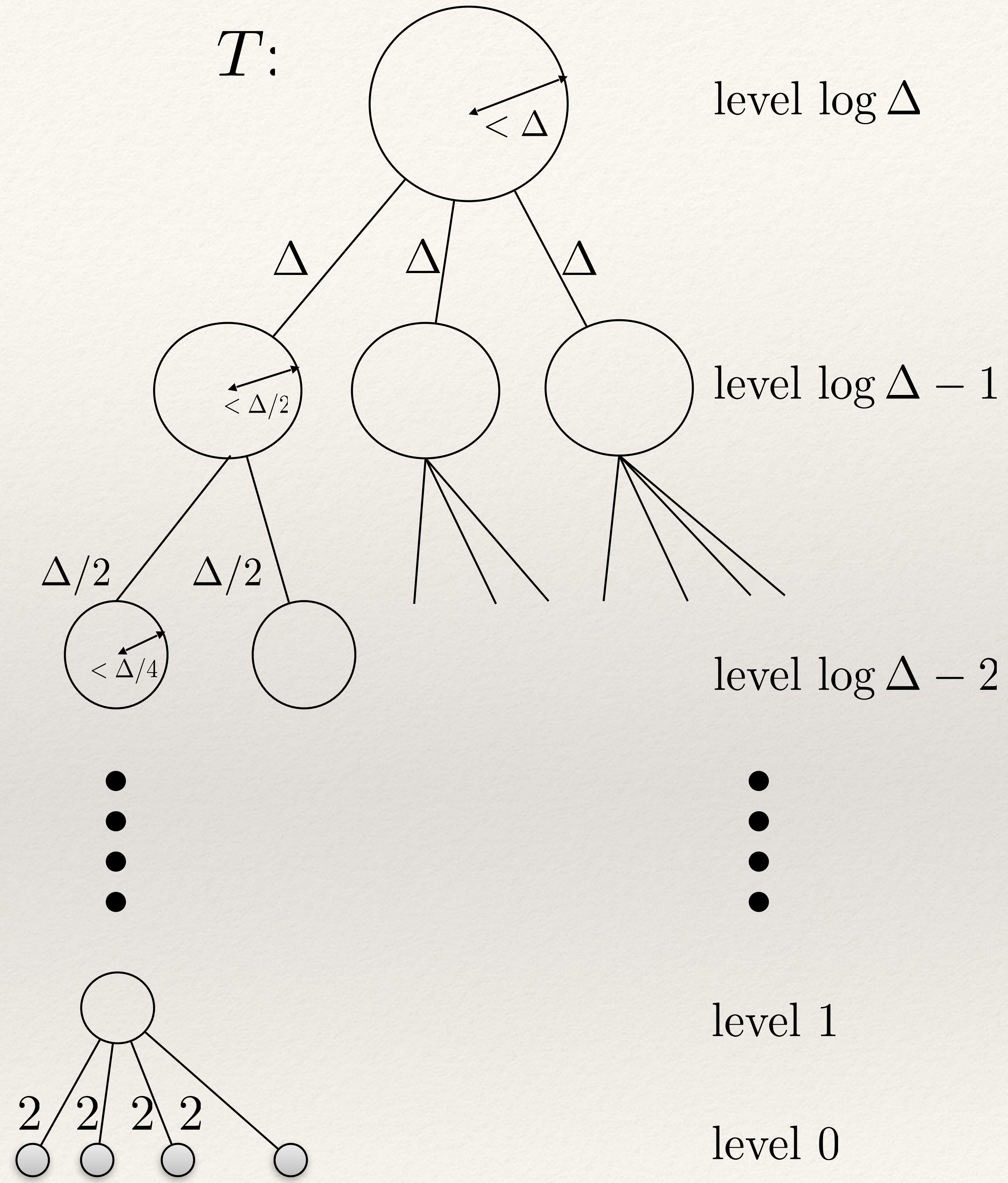
- **How to choose partitioning of G?**
- **How to choose distances in T?**

Choosing tree distances

- W.l.o.g. $d_{u,v} \geq 1$
- Δ = smallest power of 2 greater than $2 \cdot \max d_{u,v}$
- FRT decomposition guarantee: for a set S on level i, there exists a ball of radius $< 2^i$ (centered at some node) containing nodes of S.



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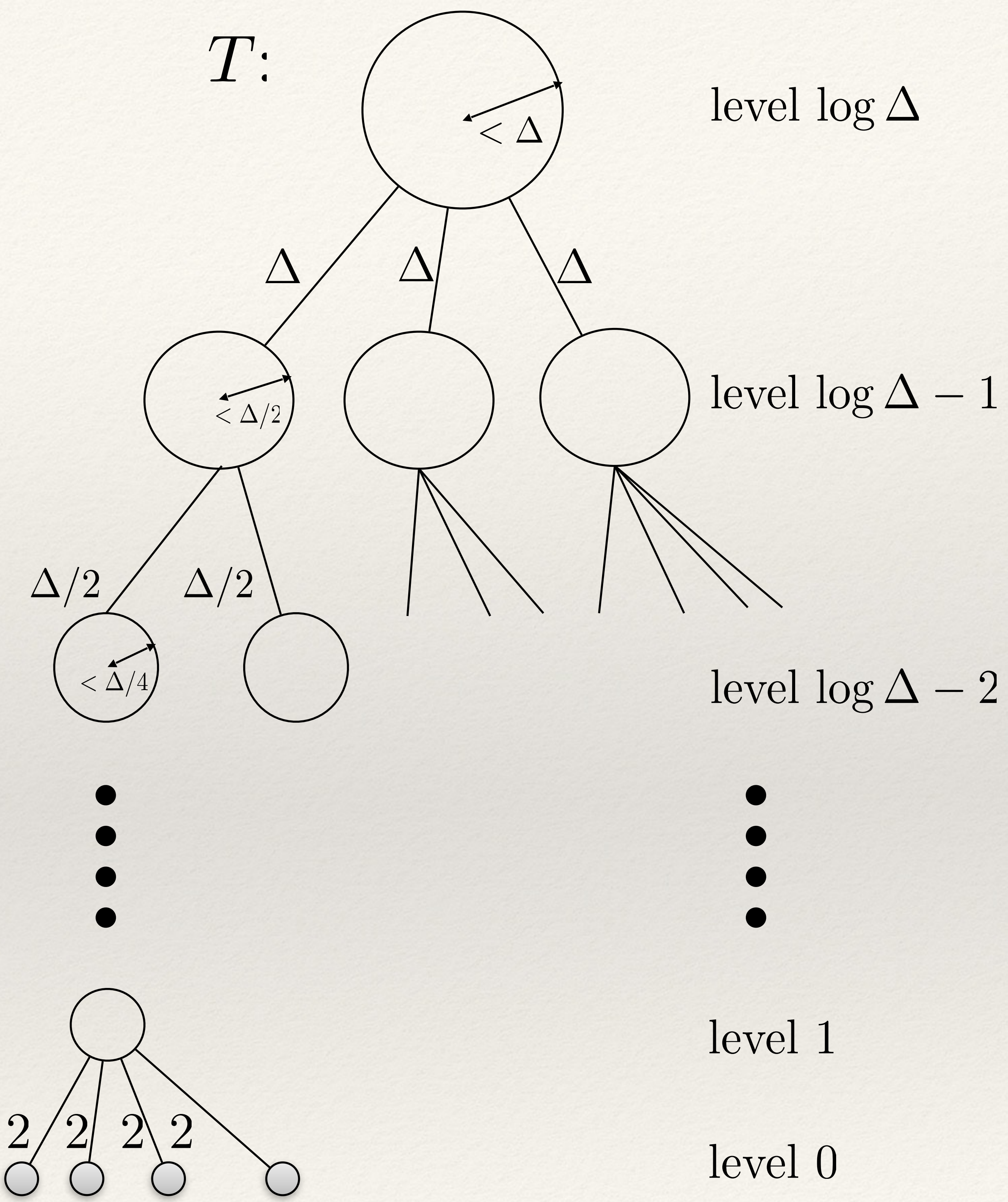


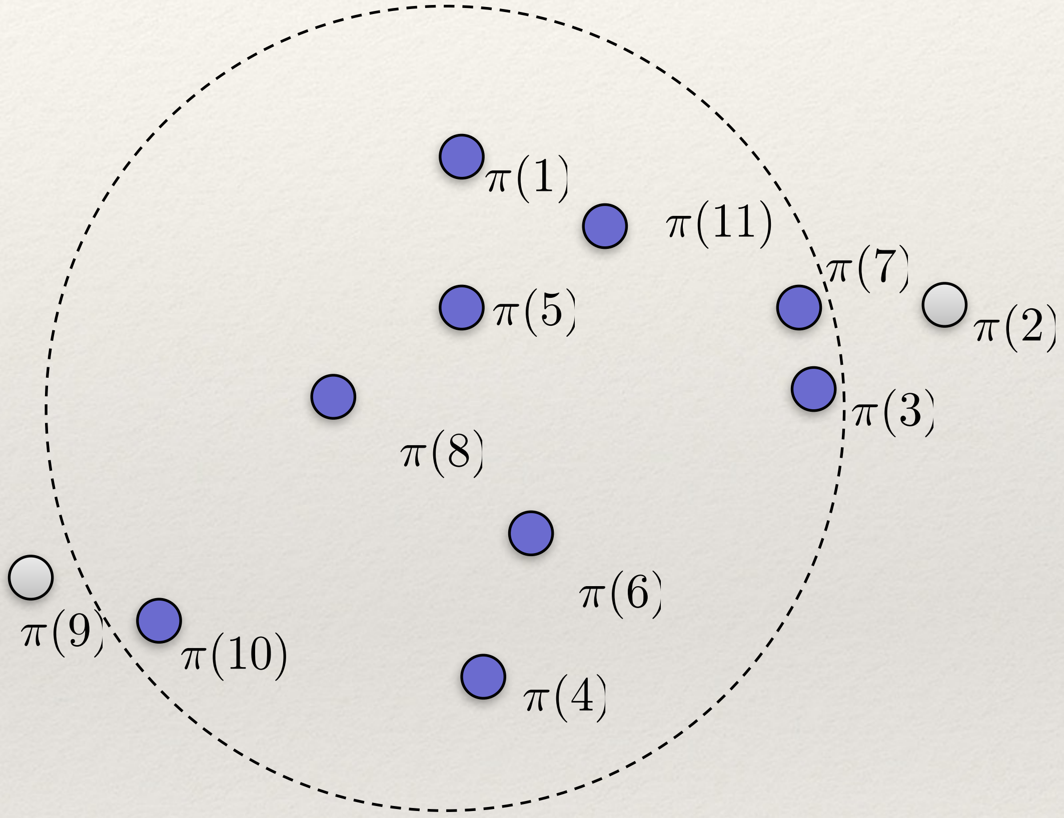
Observation. For nodes $u, v \in V$, s.t. lca(u,v) is on level i it holds that $T_{u,v} = 2^{i+2} - 4$.

Lemma 1. $d_{u,v} \leq T_{u,v}$

Partitioning of a single set

At the beginning randomly choose $r \in [1/2, 1)$ and a random permutation π of all nodes.

Partitioning of *S* on level *i+1*:



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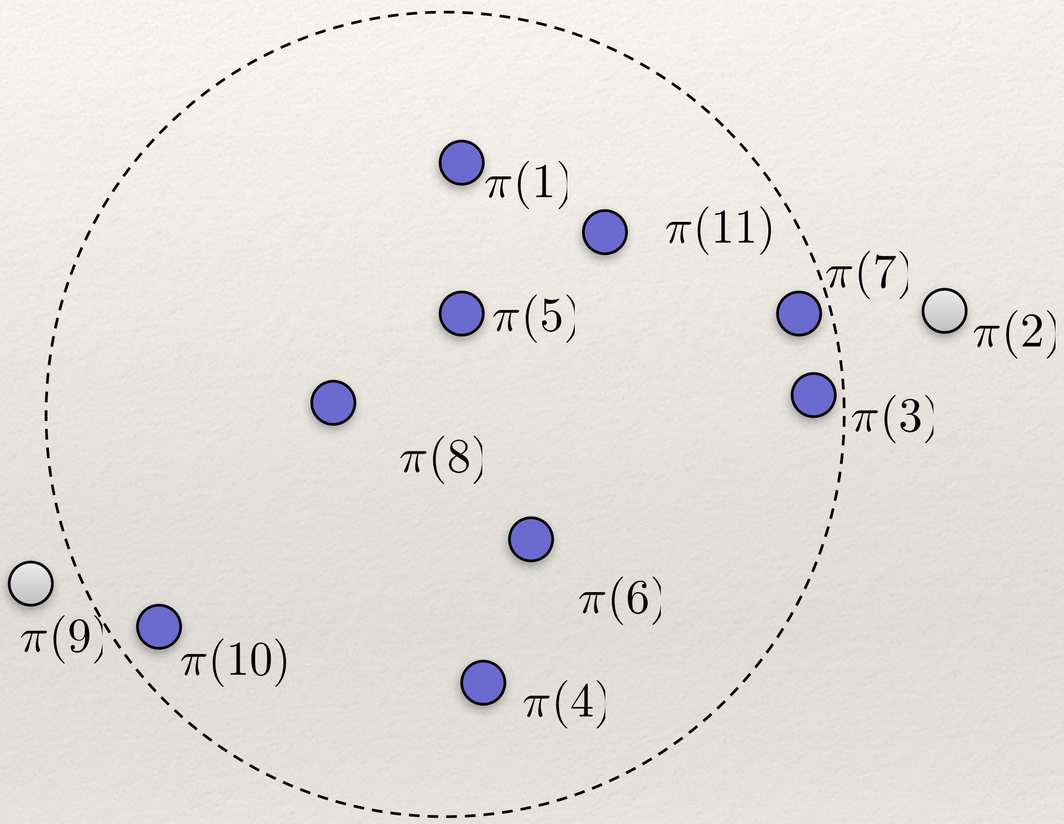

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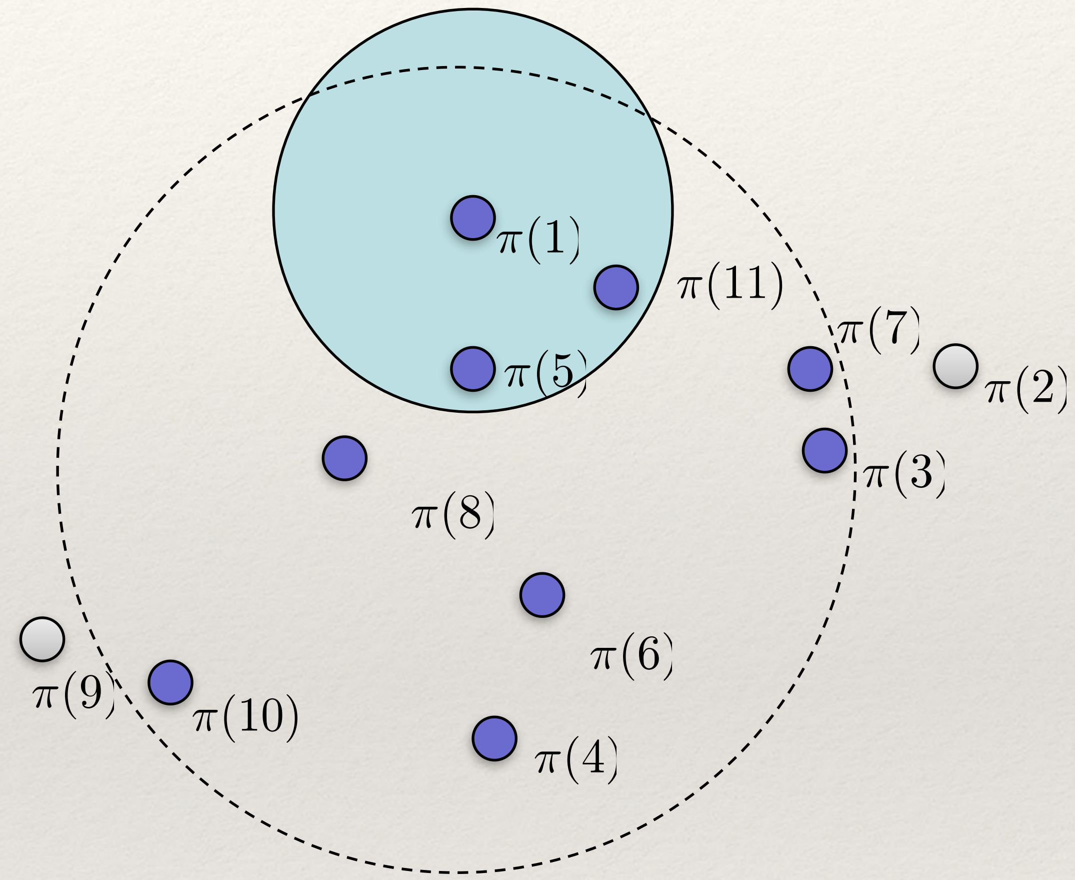

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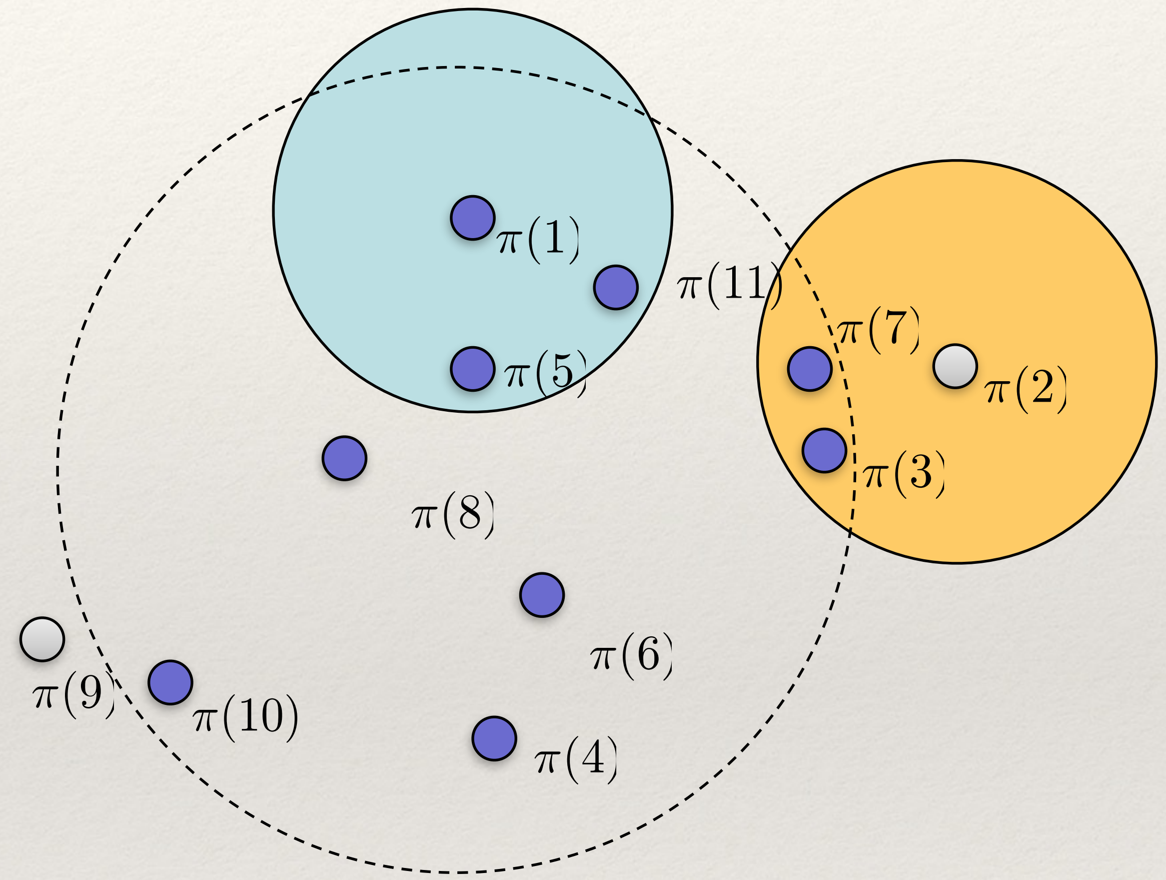

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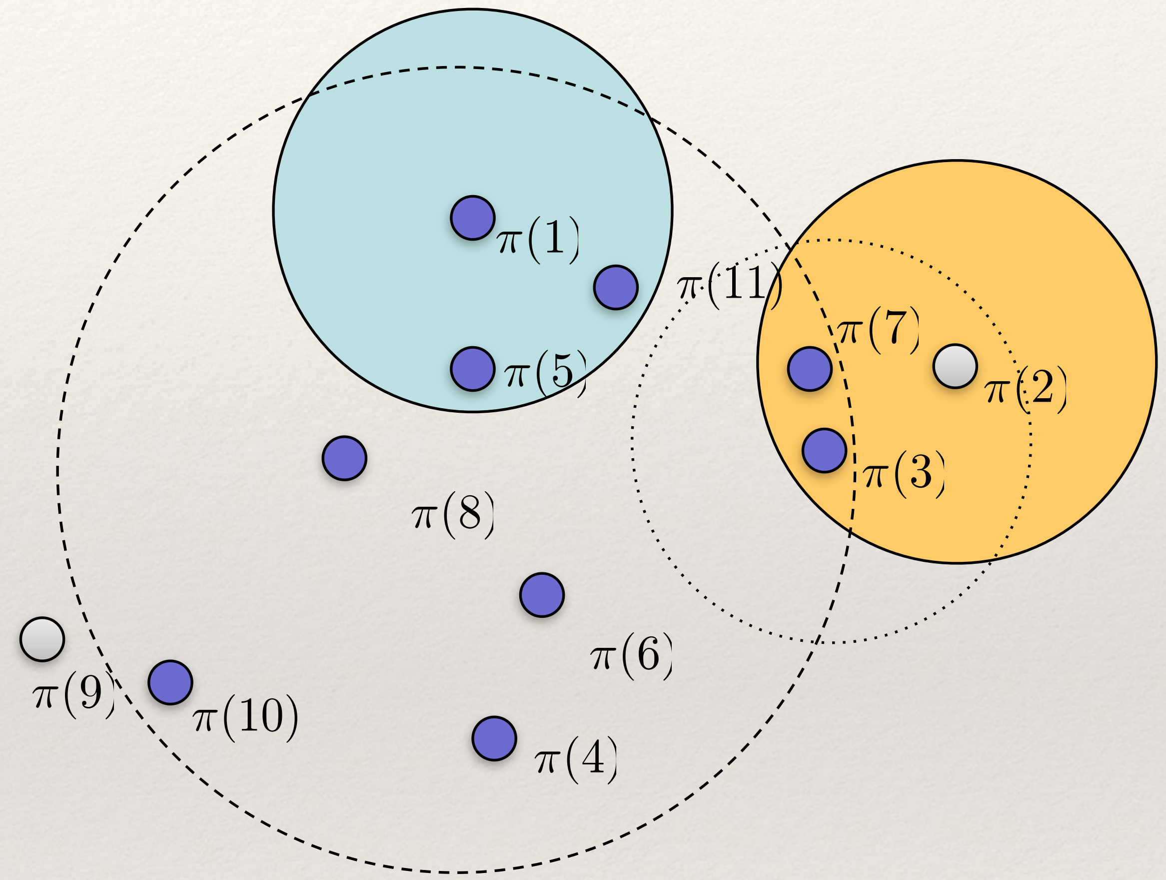

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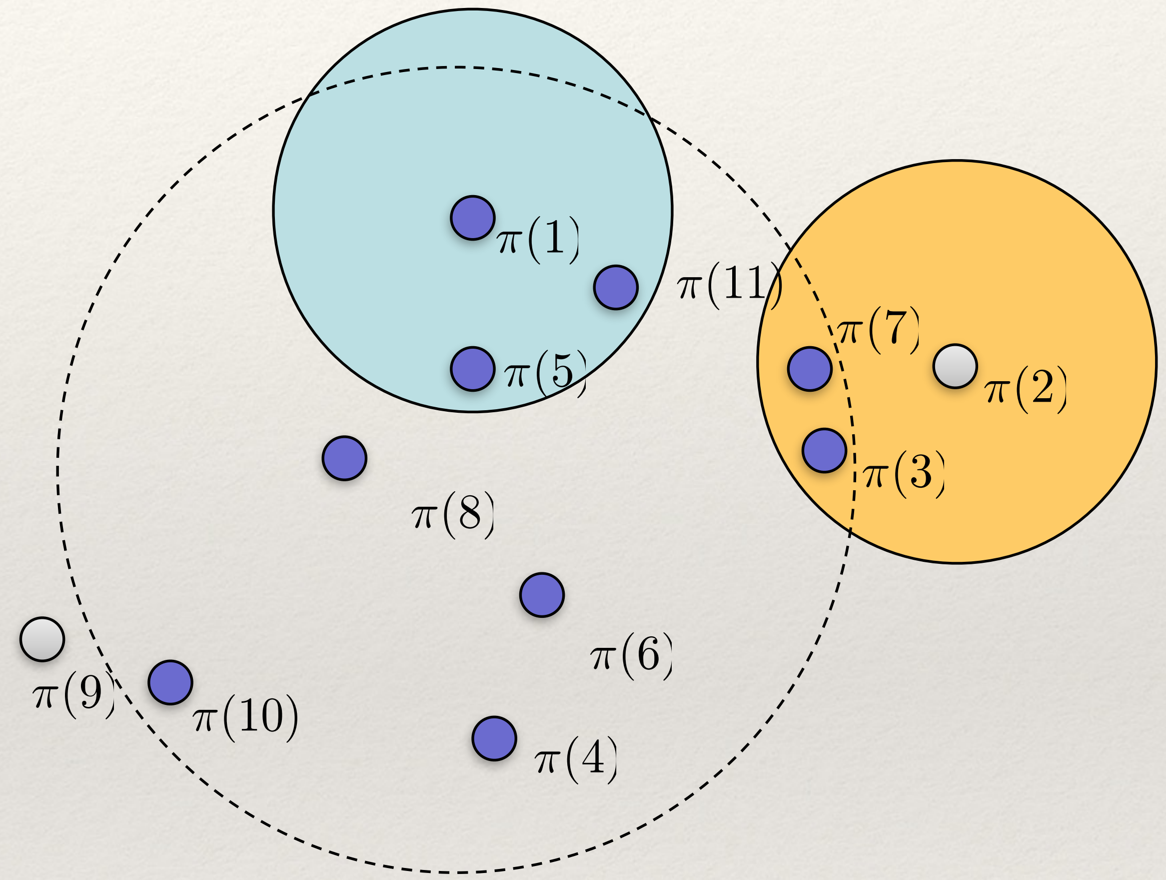

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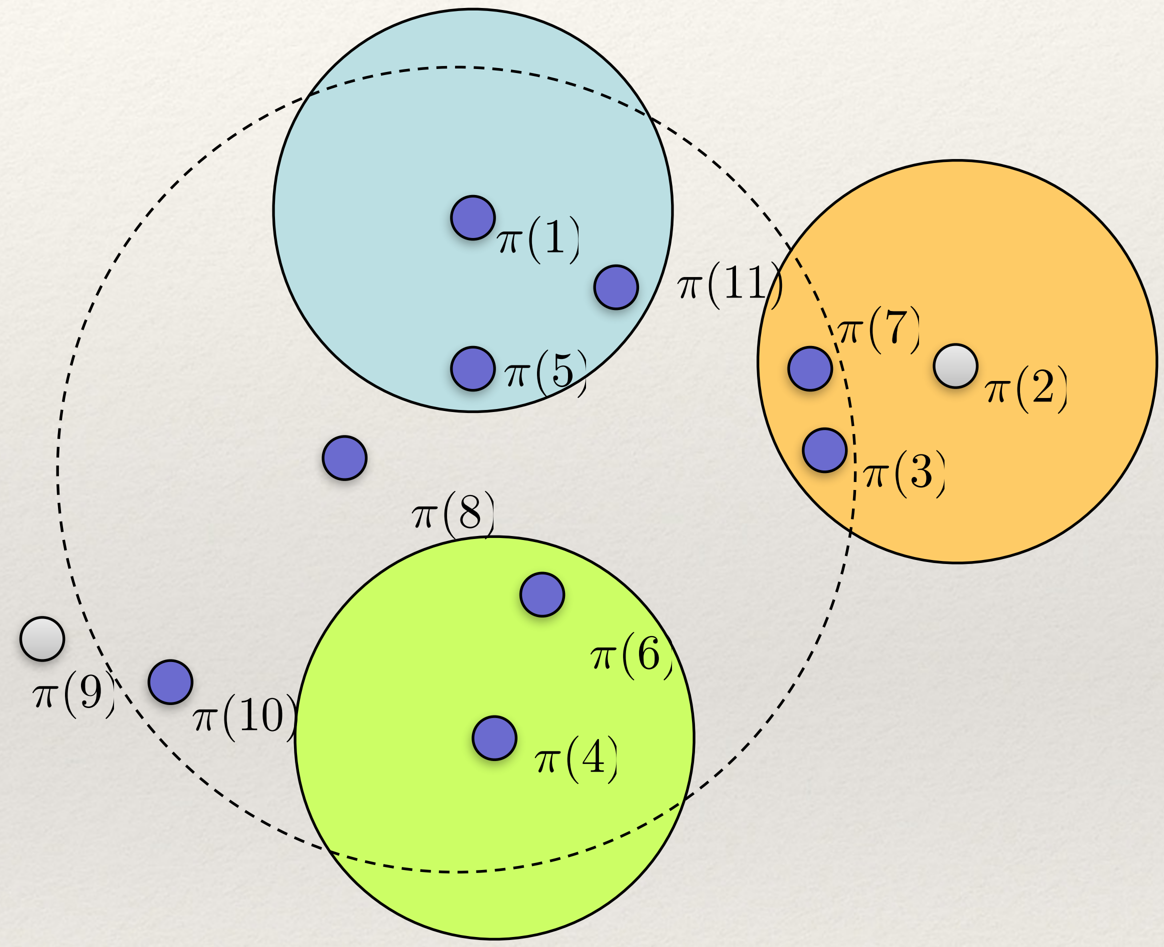

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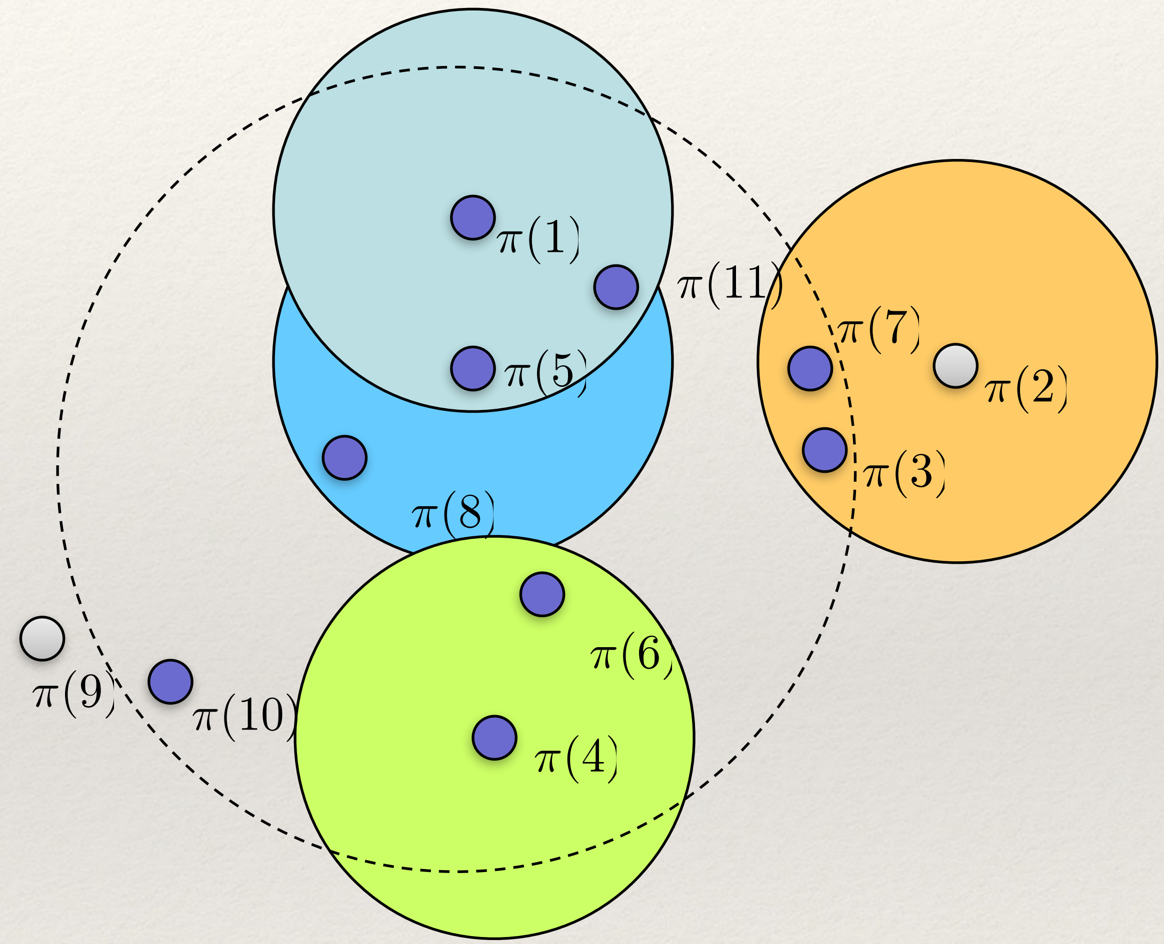

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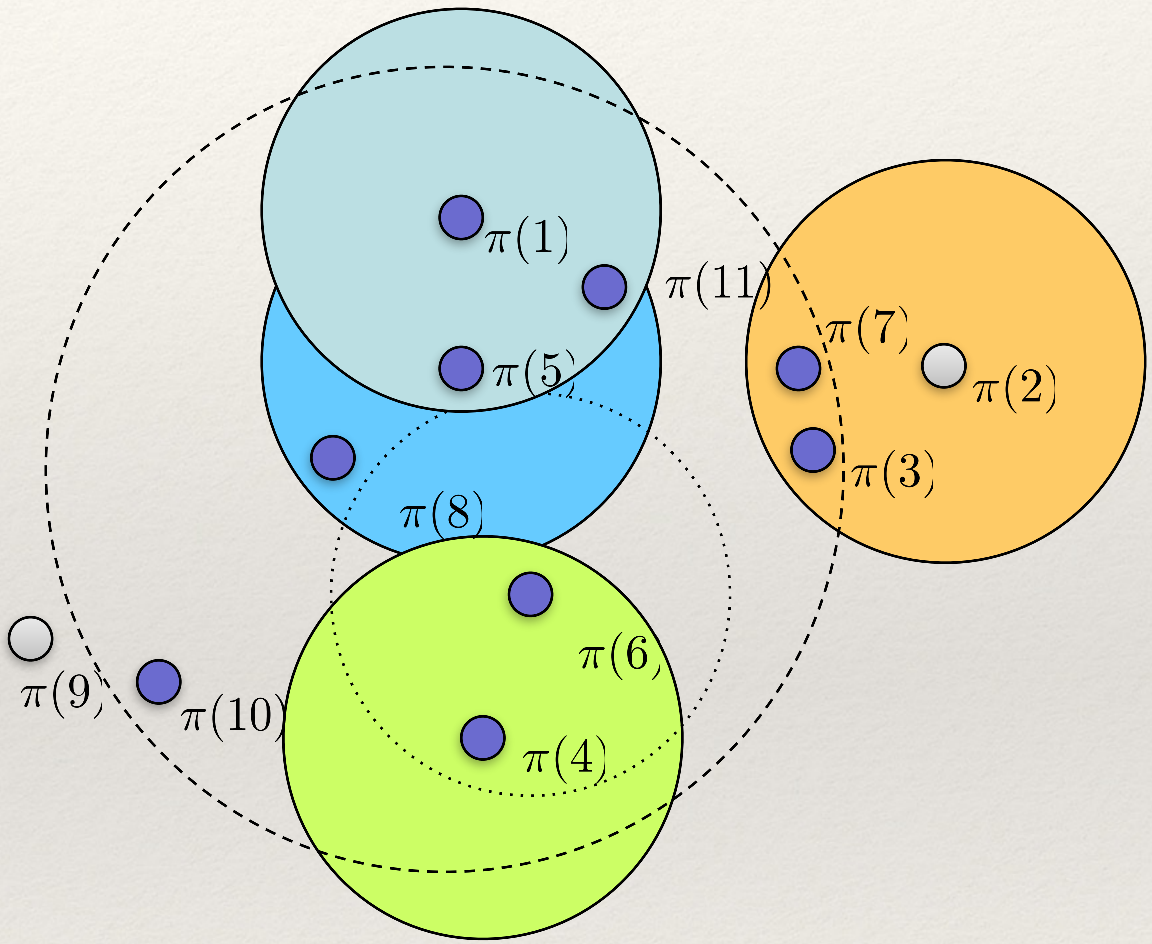

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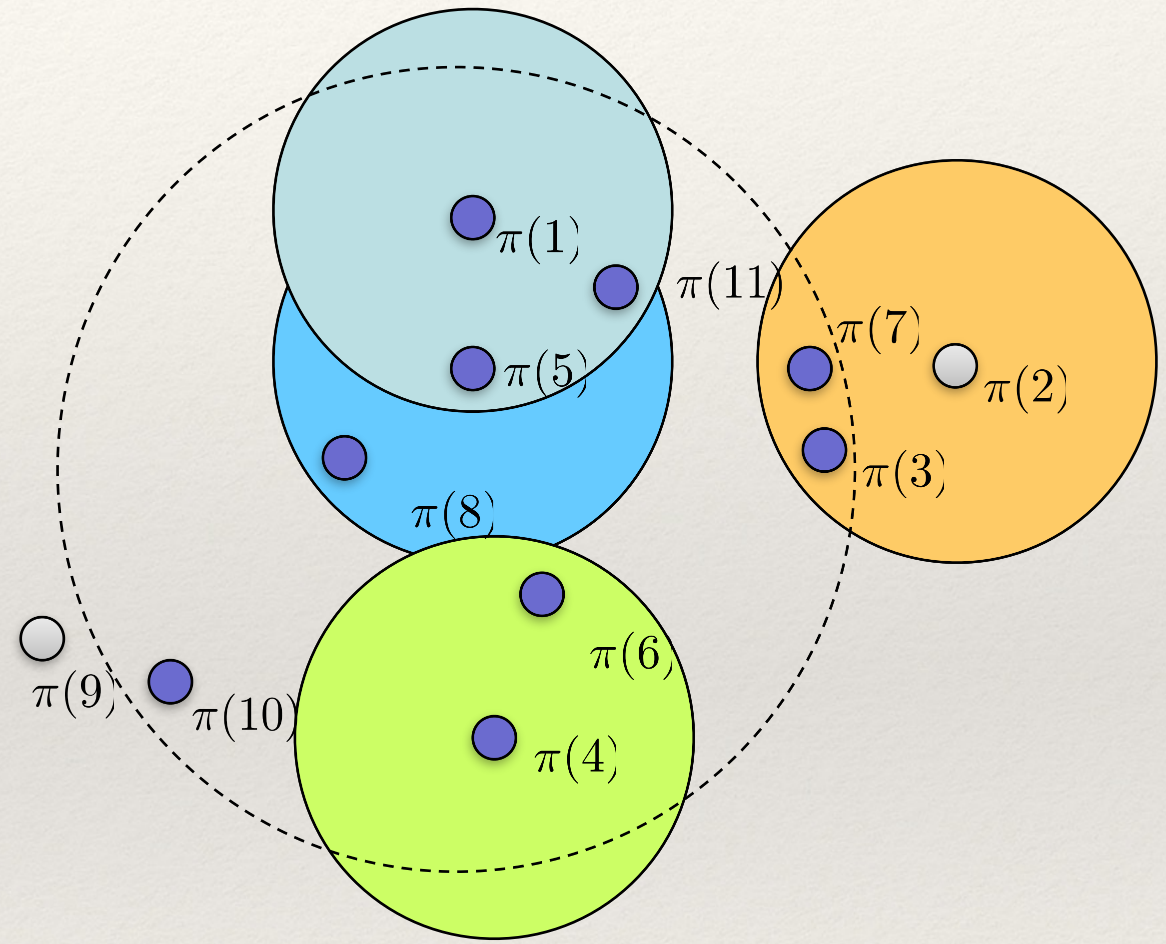

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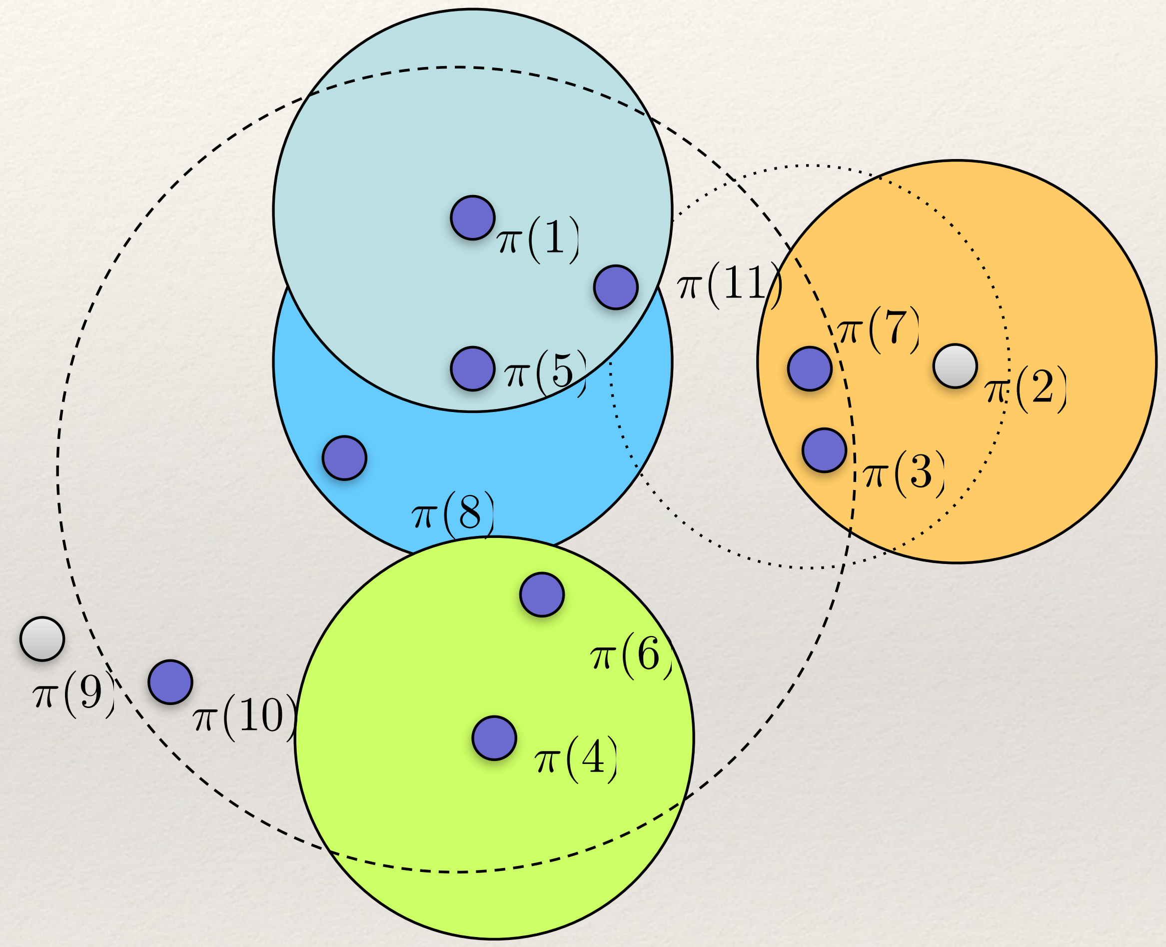

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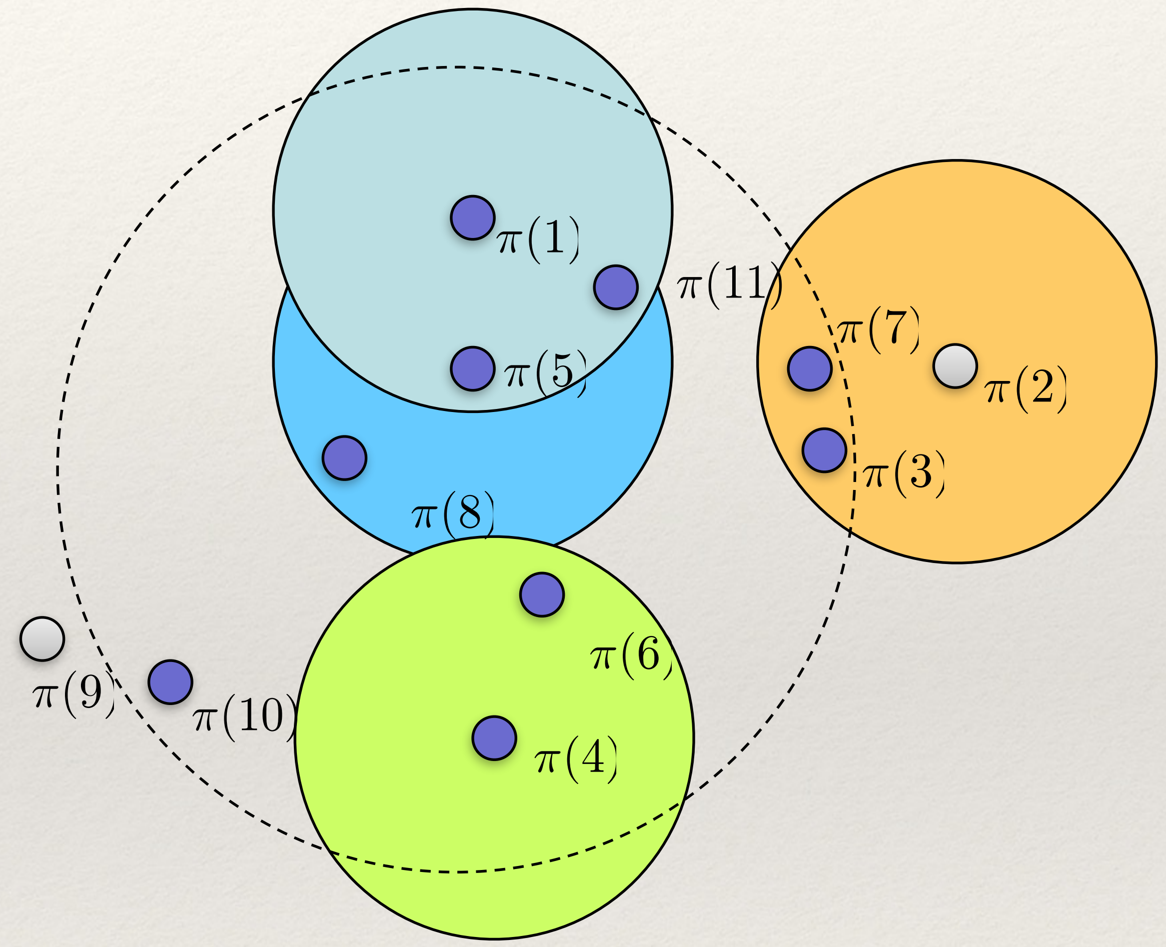

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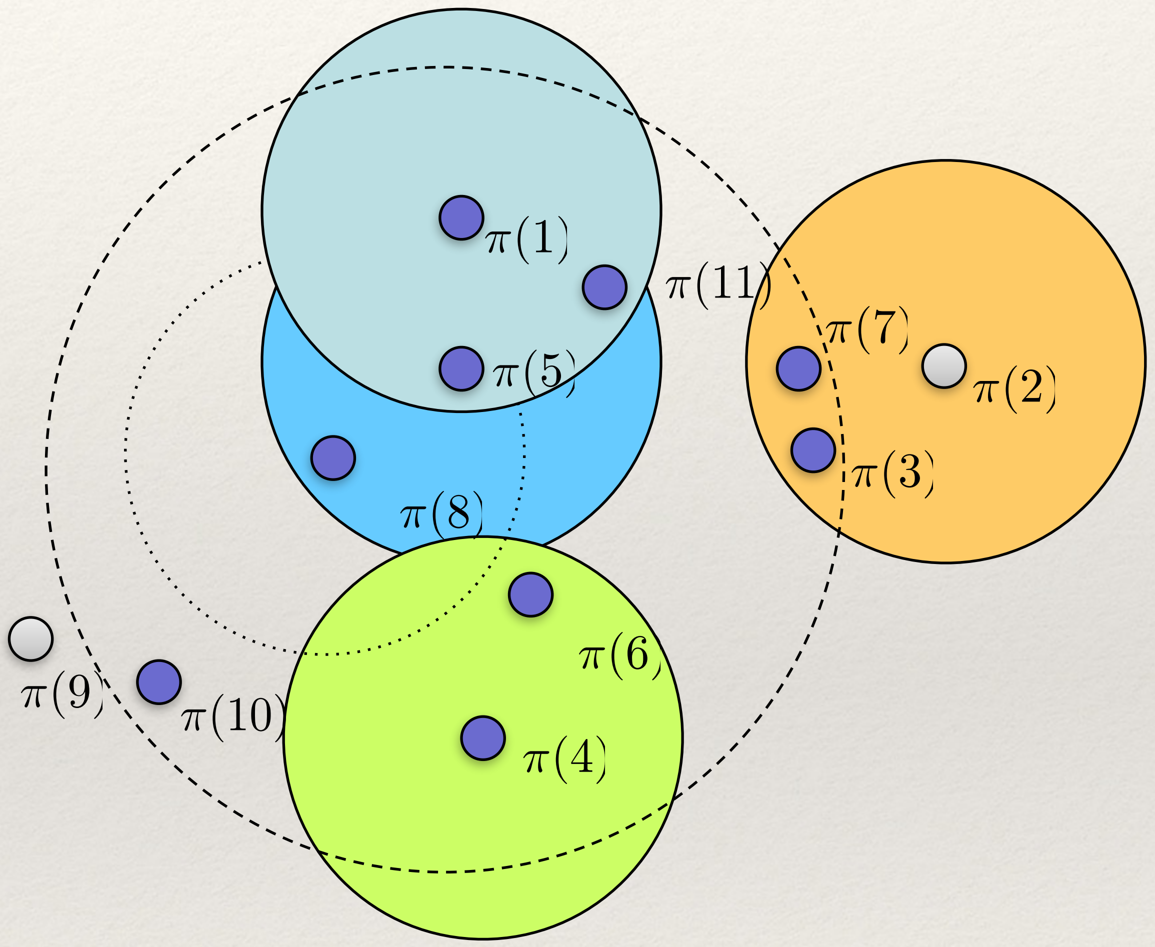

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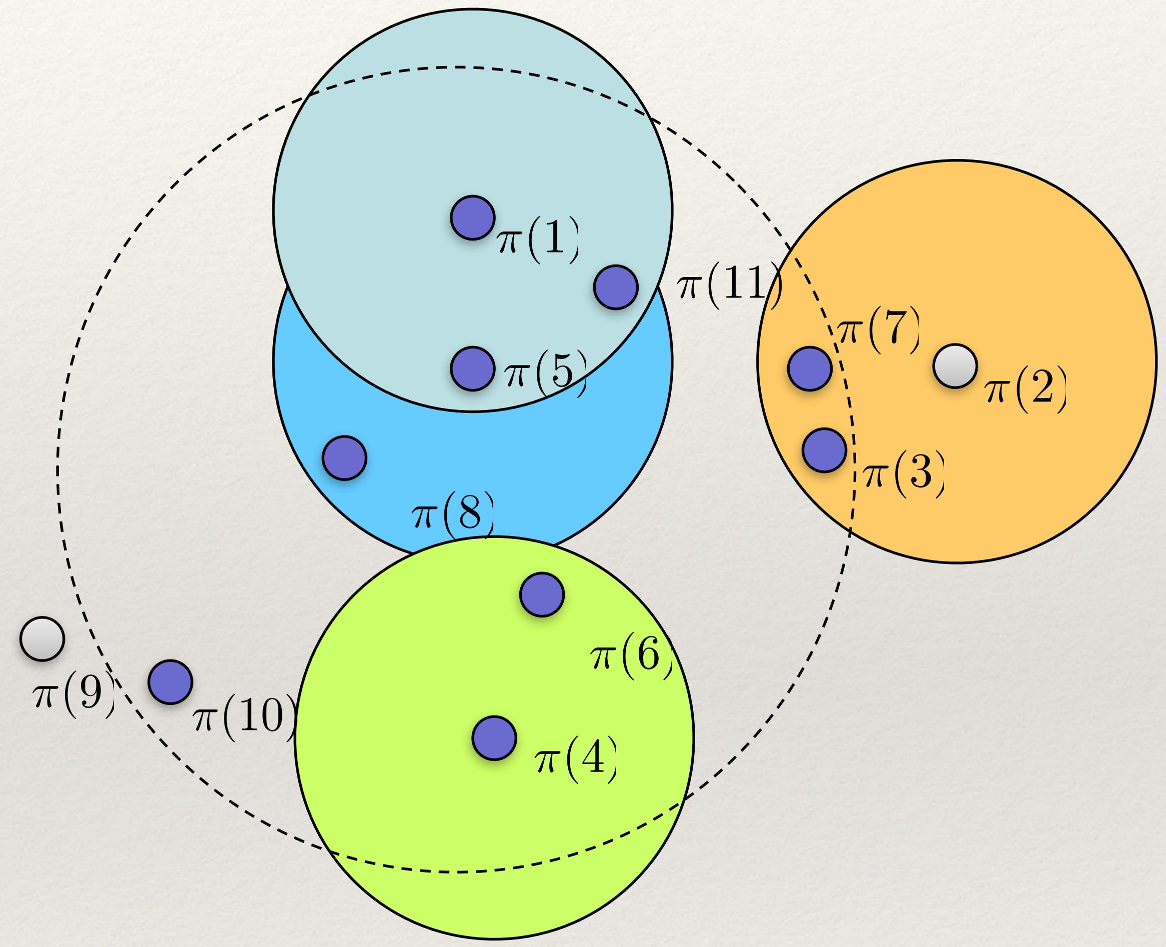

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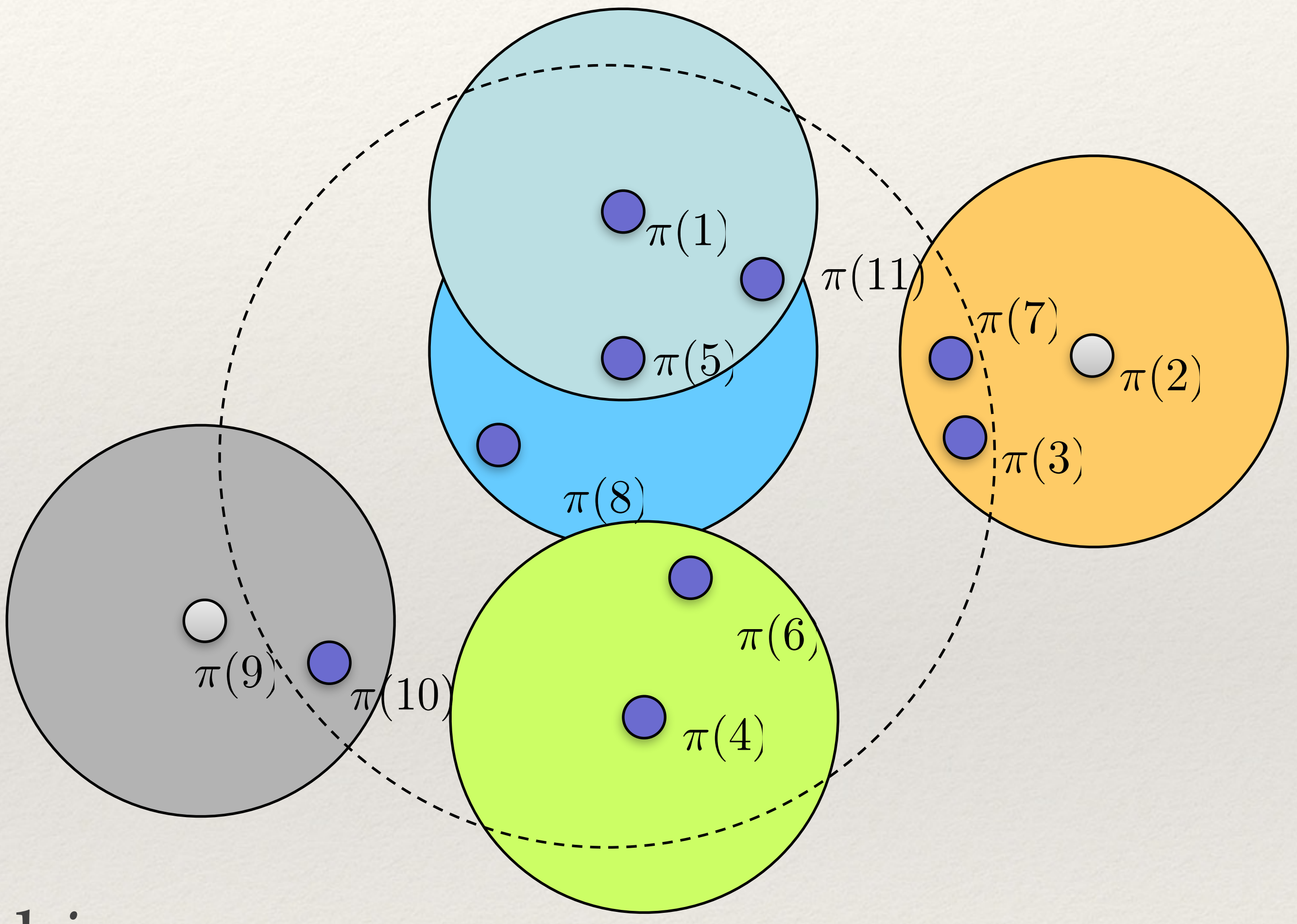

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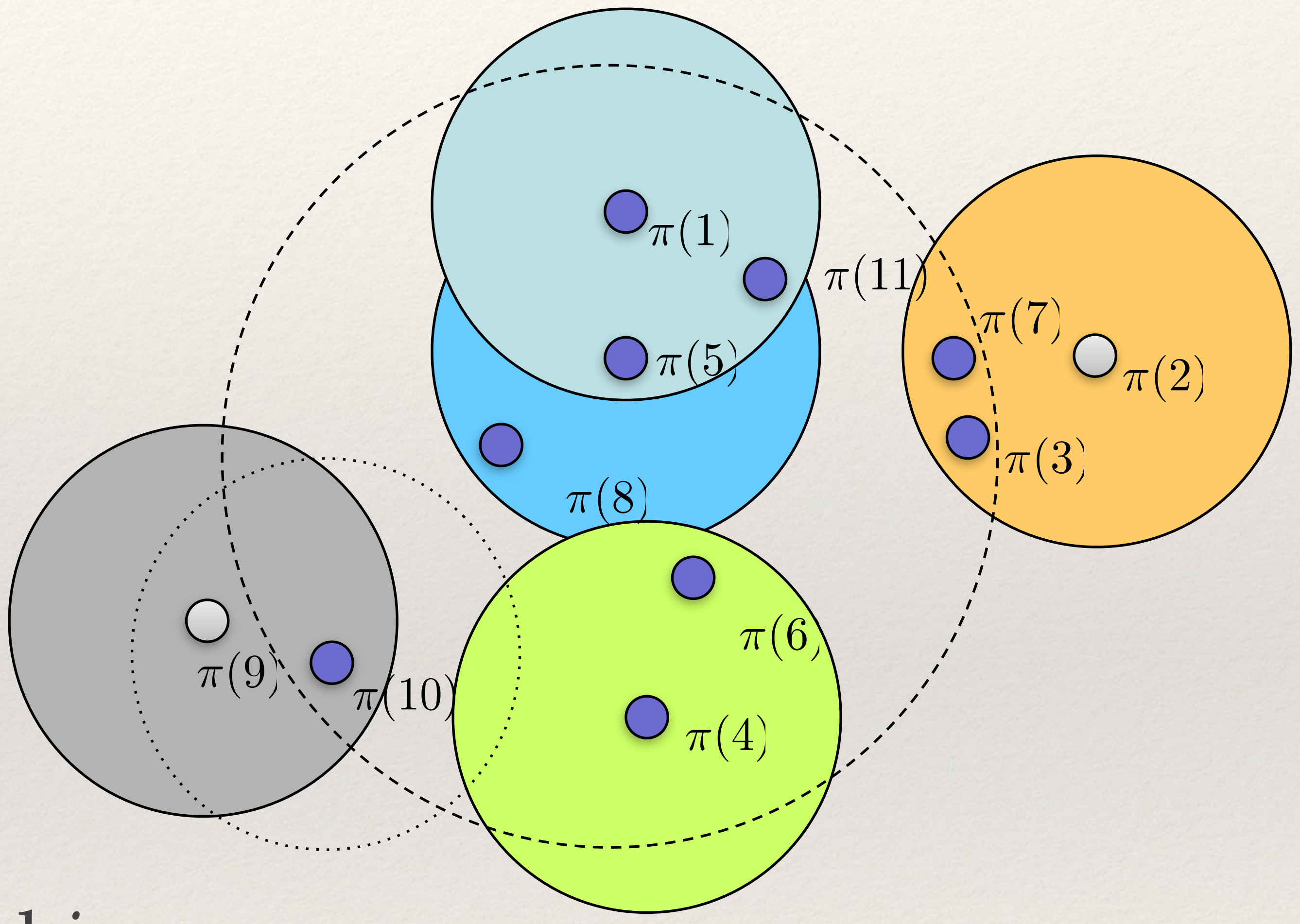

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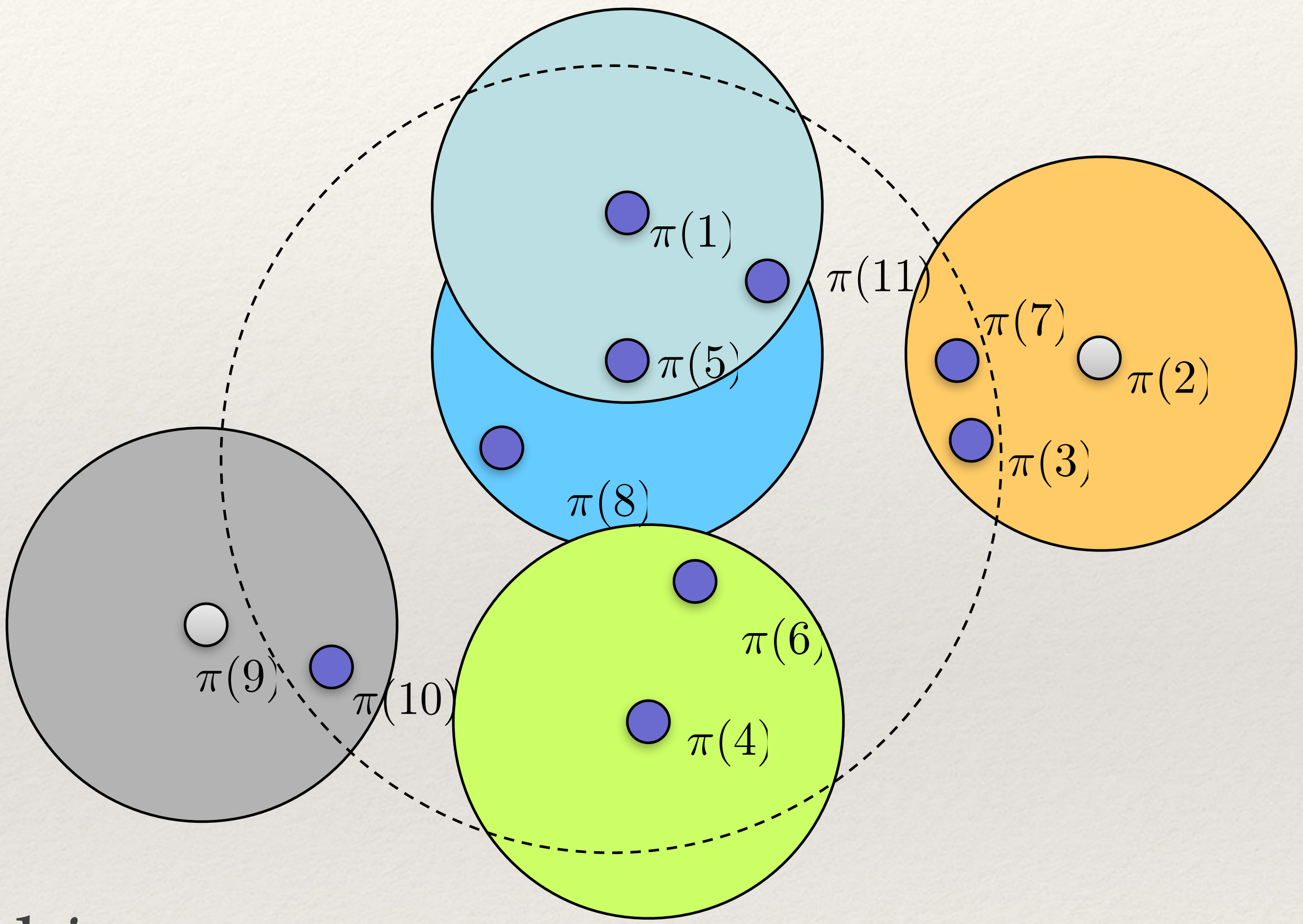

Partitioning of a single set

At the beginning randomly choose $r \in [1/2, 1)$ and a random permutation π of all nodes.

Partitioning of S on level $i+1$:

<u>for</u> $j = 1, \ldots, n$

- $X_j \leftarrow \mathrm{Ball}(\pi(j), 2^i \cdot r)$
- <u>if</u> $X_j \cap S \neq \varnothing$ <u>then</u>
  - $X_j \cap S$ is the new set on level i
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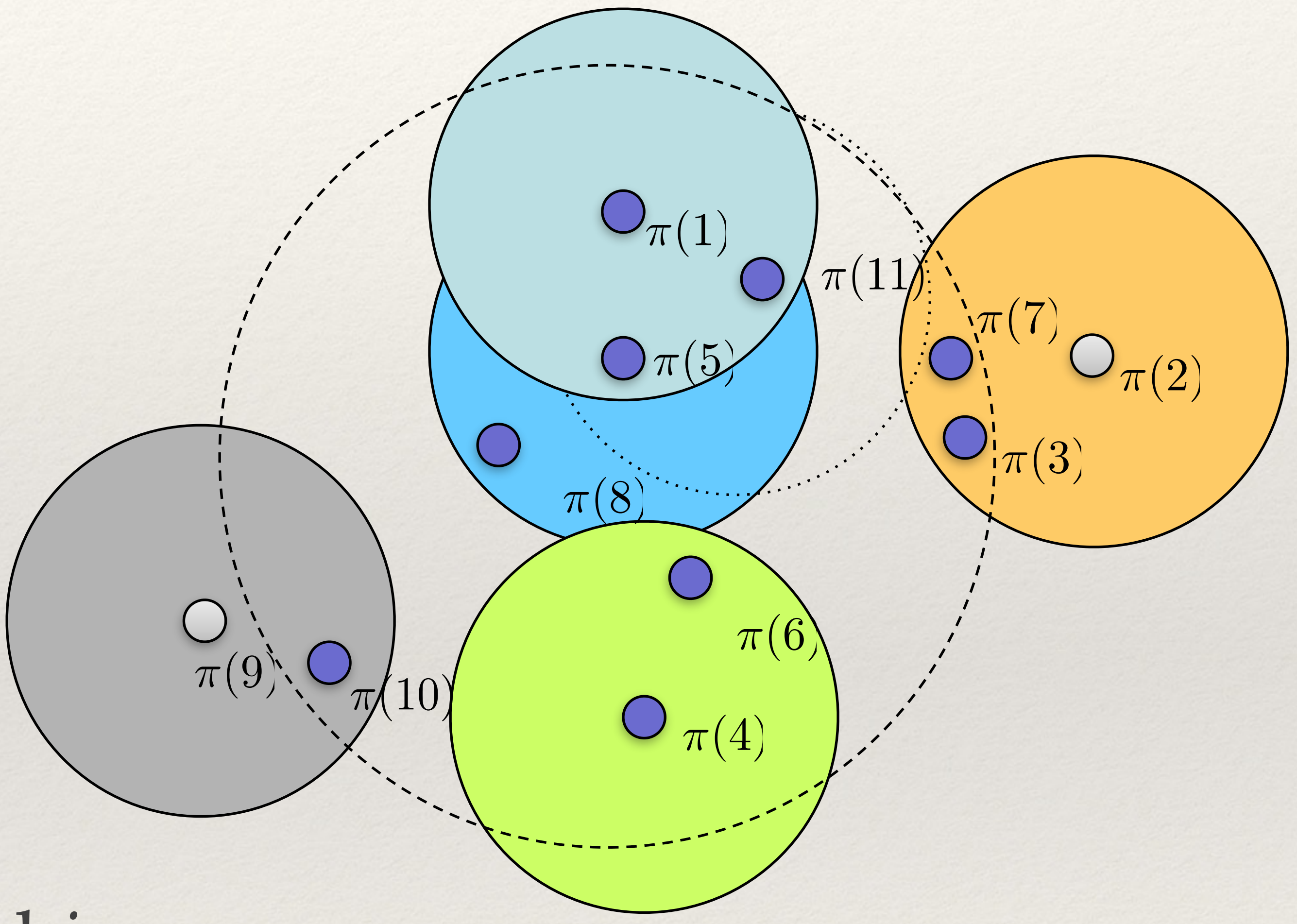

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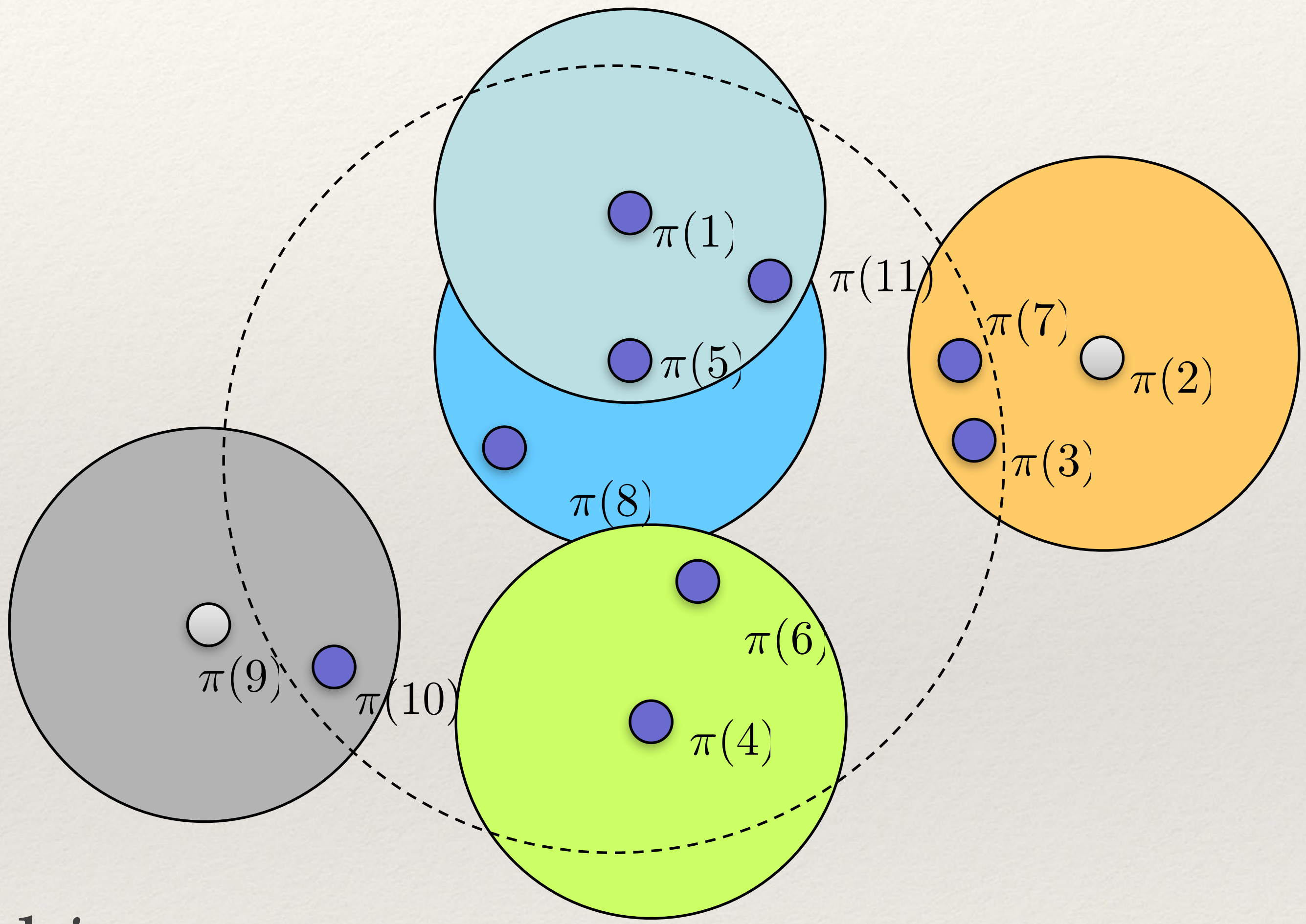
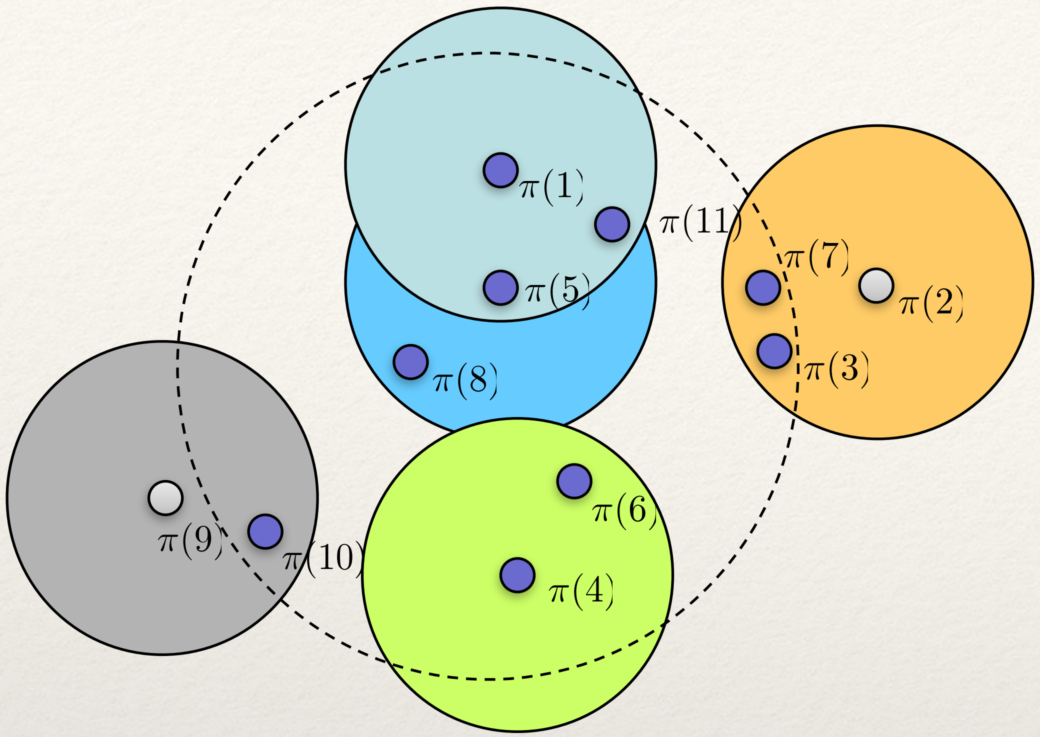
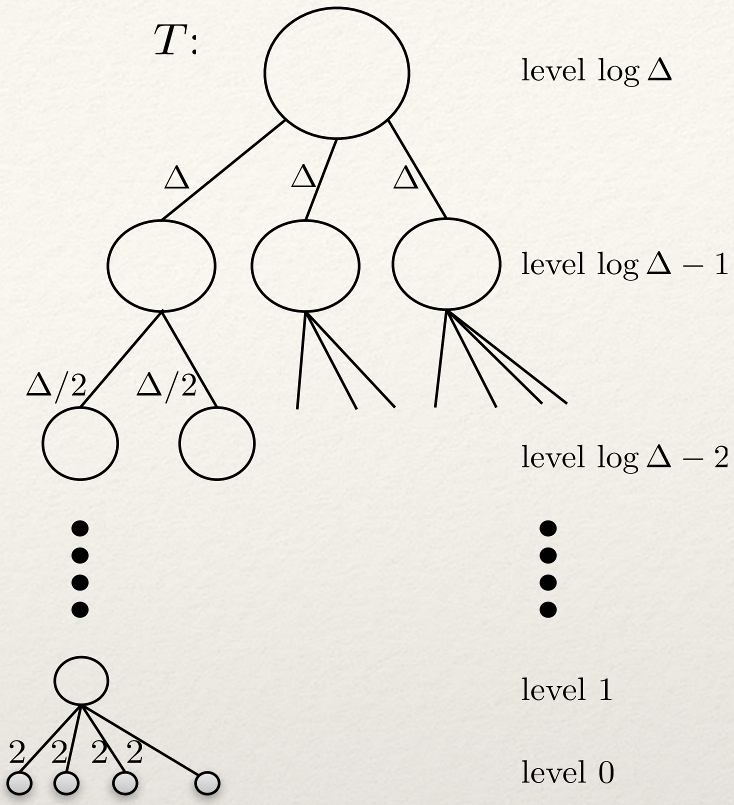


Bounding distances in tree (1)



Random variables

- $r \in [1/2, 1)$
- permutation π



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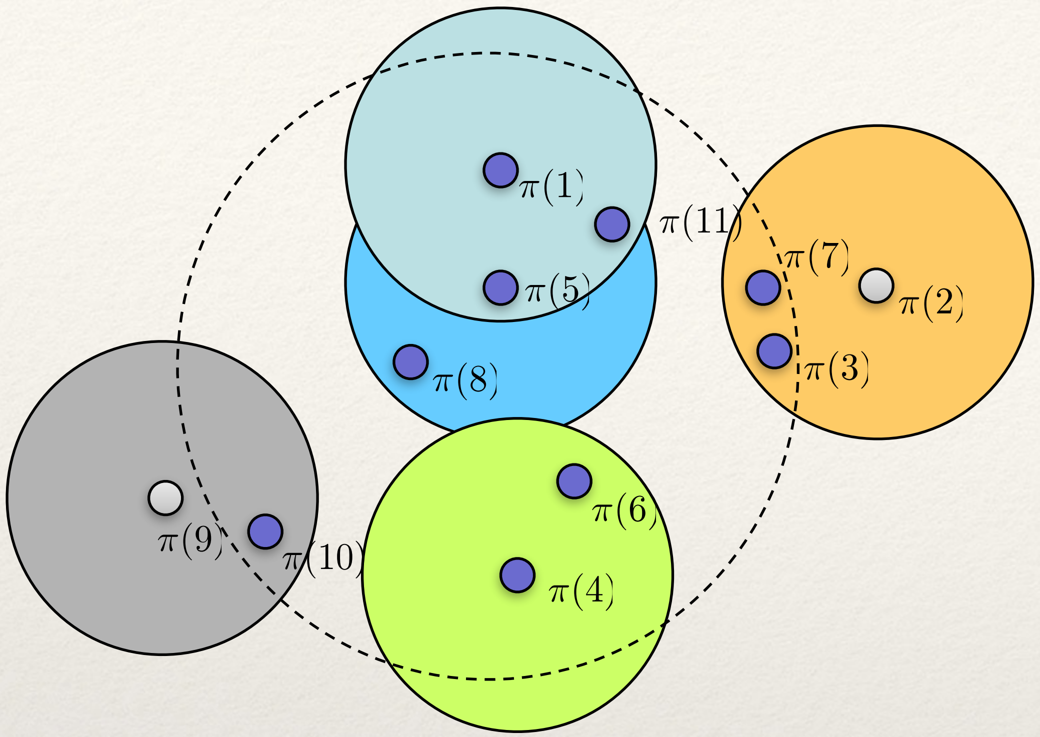

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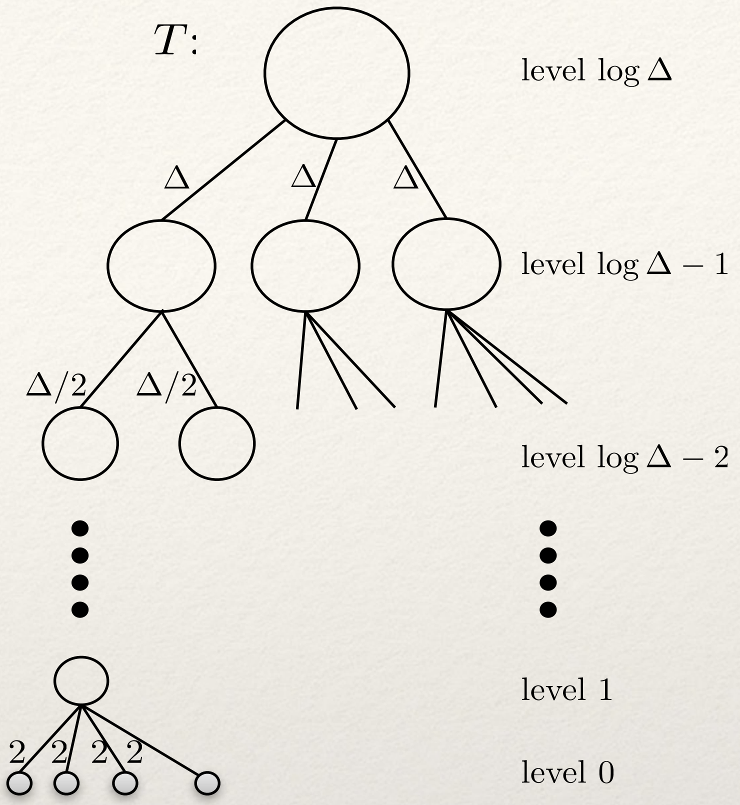

Lemma 2. $$\mathbf{E}[T_{u,v}] \leq \sum_{i=0}^{\log\Delta-1} \sum_{w \in V} \Pr\left[\mathsf{Sep}_i^{\mathrm{A}}(w) \wedge \mathsf{Sep}_i^{\mathrm{B}}(w)\right] \cdot 2^{i+3}$$

where

- $\mathsf{Sep}_i^{\mathrm{A}}(w)$ = exactly one from $\{u,v\}$ belongs to Ball$(w, 2^i \cdot r)$
- $\mathsf{Sep}_i^{\mathrm{B}}(w)$ = for all w' s.t. $\pi(w') < \pi(w)$ it holds that $u,v \notin$ Ball$(w', 2^i \cdot r)$

Bounding distances in tree (2)

- $\mathsf{Sep}_i^{\mathrm{A}}(w)$ = exactly one from *{u,v}* belongs to Ball(*w*, 2*ⁱ* · *r*)
- $\mathsf{Sep}_i^{\mathrm{B}}(w)$ = for all *w′* s.t. *π(w′)* < *π(w)* it holds that *u,v* ∉ Ball(*w′*, 2*ⁱ* · *r*)

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$$= \sum_{i=0}^{\log \Delta - 1} \sum_{w \in V} \Pr\left[\mathsf{Sep}_i^{\mathrm{B}}(w) \mid \mathsf{Sep}_i^{\mathrm{A}}(w)\right] \cdot \Pr\left[\mathsf{Sep}_i^{\mathrm{A}}(w)\right] \cdot 2^{i+3}$$

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Lemma 3. for any *i*: $$\sum_{w \in V} \Pr\left[\mathsf{Sep}_i^{\mathrm{B}}(w) \mid \mathsf{Sep}_i^{\mathrm{A}}(w)\right] \leq H_n$$

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⇓

$$\mathbf{E}[T_{u,v}] \leq 16 \cdot H_n \cdot d_{u,v} = O(\log n) \cdot d_{u,v}$$

Thank you for you attention!