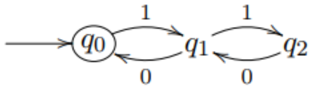
GandALF — Exercise Sheet 2

Exercise 1. Is the set of all eventually periodic words Büchi Recognizable? A word is eventually periodic if it is of the form wv^ω. *I found this exercise together with a beautifully simple solution last Friday on the /r/compsci subreddit.*

The following six exercises are from the Ong's book.

3.1 Consider the following Büchi automaton A:

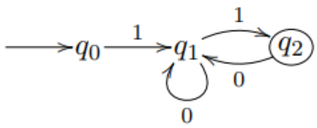


(a) Construct an Existential S1S-formula equivalent to A.

(c) A *star-free ω-regular language* over an alphabet Σ is a finite union of ω-languages of the form $U \cdot V^\omega$, where U and V are (regular) languages constructed from a finite set of finite words over Σ using the Boolean operations, namely, complementation, union and concatenation.

Prove that A recognises a star-free regular ω-language.

3.2 Let A be the following Büchi automaton A:



Construct an S1S-formula $\varphi(X)$ such that $\alpha \in \mathbb{B}^\omega$ satisfies φ iff A accepts α.

3.4 Give S1S-formulas $\varphi_1(X_1, X_2)$ and $\varphi_2(X_1, X_2)$ for the following ω-languages:

(a) $L_1 = \binom{1}{1}\binom{1}{0}^*\begin{pmatrix}1\,0\\1\,0\end{pmatrix}^\omega$

(b) $L_2 = \begin{pmatrix}1\,1\,1\\1\,1\,1\end{pmatrix}\binom{0}{1}^\omega$

Explain the purpose of the main subformulas of $\varphi_1(X_1, X_2)$ and $\varphi_2(X_1, X_2)$.

3.5 Show that (natural numbers) addition $x = y + z$ is not definable in S1S.

[*Hint.* Show that S1S-definability of addition would imply that the language $\{ a^n b^n c^\omega : n \geq 0 \}$ is Büchi recognizable.]

3.6 *Presburger arithmetic* is first-order logic over the structure $(\omega, +)$ where

$$+ \; = \; \{ (a, b, c) \in \omega^3 : a + b = c \}.$$

A number can be represented as a finite set of numbers corresponding to the positions of 1s in its binary representation.

(a) Show that there is an S1S formula $\varphi(X, Y, Z)$ asserting that the numbers a, b and c represented respectively by the finite sets X, Y and Z are related by the equation $a+b = c$.

[*Hint.* Recall the idea of full adder in Digital Hardware.]

(b) Deduce that formulas of Presburger arithmetic can be translated into S1S.

Hence prove that Presburger arithmetic is decidable.

Why does this not contradict the preceding question?

3.7 *Weak monadic second-order theory of one successor*, WS1S, is defined in the same way as S1S except that second-order variables range over only finite sets of natural numbers.

(a) Fix a deterministic Muller automaton A. Since it is not possible to say anything in WS1S about any complete run directly, we restrict ourselves to prefixes of runs. Note that every ω-word that is accepted by a deterministic Muller automaton has a *unique* accepting run.

Give a WS1S-formula that defines the ω-language recognized by A.

(b) Hence deduce that an ω-language is S1S-definable iff it is WS1S-definable.

Exercise 8. 2-satisfiability is the problem of determining whether a boolean formula in conjunctive normal form with two variables per clause is satisfiable. Prove that 2-satisfiability is NL-complete.