# GandALF: mean-payoff games

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# Definitions

- Game arena: $A = (V_1, V_2, E)$ (no dead ends).
- Play on A: $\pi = v_0 v_1 \ldots$; All plays $\Pi$.
- Objective (payoff function): $f\colon \Pi \to \mathbb{R}$
- Mean-payoff objective: labeling $w\colon E \to \{-W, \ldots, W\}$ gives $MP(\pi) = \liminf_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} w(\pi[i], \pi[i+1])$.
- Mean-payoff game: $G = (A, w)$.
- $G[\sigma_1, \sigma_2, v]$ play starting from v consistent with $\sigma_1, \sigma_2$.
- Values $val^- \geq val^+$ defined as:

$$val^-(v) = \inf_{\sigma_1 \in S_1} \sup_{\sigma_2 \in S_2} MP(G[\sigma_1, \sigma_2, v])$$

$$val^+(v) = \sup_{\sigma_2 \in S_2} \inf_{\sigma_1 \in S_1} MP(G[\sigma_1, \sigma_2, v])$$

- Threshold objectives.

# Memoryless determinacy

**Theorem (Memoryless determinacy of mean-payoff games)**

Let $\mathrm{G} = (\mathrm{A}, \mathrm{w})$ be a mean-payoff game.
(1) For all $\mathrm{v} \in \mathrm{A}$, we have $\mathrm{val}^-(\mathrm{G}, \mathrm{v}) = \mathrm{val}^+(\mathrm{G}, \mathrm{v})$.
(2) Each player has a memoryless strategy to ensure (uniformly in v) $\mathrm{val}(\mathrm{G}, \mathrm{v})$: There exist memoryless $\sigma_1^*, \sigma_2^*$ such that

$$\sup_{\sigma_2 \in \mathrm{S}_2} \mathrm{MP}(\mathrm{G}[\sigma_1^*, \sigma_2, \mathrm{v}]) = \mathrm{val}(\mathrm{G}, \mathrm{v}) = \inf_{\sigma_1 \in \mathrm{S}_1} \mathrm{MP}(\mathrm{G}[\sigma_1, \sigma_2^*, \mathrm{v}])$$

# Proof of memoryless determinacy (1)

- FMP — First-Cycle Mean-payoff games: play until a cycle is formed.
- Play $\pi = v_1 \ldots v_{k-1} v_k v_{k+1} \ldots v_{k+m}$, where $v_k = v_{k+m}$ and $v_1, \ldots, v_{k+m-1}$ are distinct.
- $\mathrm{FMP}(\pi) = \frac{1}{m} \sum_{i=0}^{m-1} w(\pi[k+i], \pi[k+i+1])$.
- Values $\mathrm{val}_F^-, \mathrm{val}_F^+$ defined accordingly.
- FMP are finite games — determined: $\mathrm{val}_F^-(v) = \mathrm{val}_F^+(v)$ (hence $\mathrm{val}_F(v)$).

**Lemma**

For all v we have $\mathrm{val}^-(v) = \mathrm{val}_F(v) = \mathrm{val}^+(v)$.

# Proof of memoryless determinacy (2)

- R-FMP(u) — First-Cycle Mean-payoff games with reset u: first occurrence of u erases memory; otherwise as FMP.
- Value $\mathrm{val}_{\mathrm{R,u}}$ defined accordingly.

**Lemma**

For all v we have $\mathrm{val}_{\mathrm{R,u}}(\mathrm{v}) = \mathrm{val}_{\mathrm{F}}(\mathrm{v})$.

**Remark**

Optimal strategy in R-FMP(u) can be made memoryless in u.

# Proof of memoryless determinacy (3)

**Lemma**

FMP enjoy memoryless determinacy.

# Computing the game value

- BDS — Bounded-duration sum games: play k rounds.
- Play $\pi = v_1 \ldots v_{k-1} v_k$.
- $BDS(\pi) = \sum_{i=0}^{k-1} w(\pi[i], \pi[i+1])$.
- Value $val_k(v)$.

**Lemma**

For all v we have $k \cdot val(v) - 2nW \leq val_k(v) \leq k \cdot val(v) + 2nW$.

**Theorem**

Mean-payoff games can be solved in $O(|V|^3|E|W)$.

Pick $k = 3n^3W$.

# Computing winning strategies

**Theorem**

Optimal memoryless strategies in mean-payoff games can be computed in $O(|V|^4|E|W\log(\frac{|E|}{|V|}))$.

# Parity games reduce to mean-payoff games

**Theorem**

Parity games with parities $\{0, \ldots, d\}$ reduce to mean-payoff games over the same arena with weights $\{-n^{d-1}, \ldots, n^{d-1}\}$.

# Discounted-sum games

- Discounted-sum objective: labeling $w\colon E \to \mathbb{Z}$ defines $\mathrm{Disc}_\lambda(\pi) = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i w(\pi[i], \pi[i+1])$.
- Values $\mathrm{val}_\lambda^-(v), \mathrm{val}_\lambda^+(v)$.

**Theorem**

The values of a discounted-sum game $(A, w, \lambda)$ are the unique solutions of the following equations:

$$x_i = \min_{(v_i,v_j)\in E}(1-\lambda)w(v_i, v_j) + \lambda x_j \quad \text{if } x_i \in V_1$$

$$x_i = \max_{(v_i,v_j)\in E}(1-\lambda)w(v_i, v_j) + \lambda x_j \quad \text{if } x_i \in V_2$$

# Mean-payoff to discounted-sum games

### Theorem

Mean-payoff games reduce to discounted-sum games. The reduction preserves optimal strategies.

### Lemma

Consider game arena A and labeling w.

$$\mathrm{val}(v) - 2n(1-\lambda)W \leq \mathrm{val}_\lambda(v) \leq \mathrm{val}(v) + 2n(1-\lambda)W$$

# Discounted-sum games in $\mathrm{UP} \cap \mathrm{coUP}$

**Theorem**

The unique solutions of the following equations corresponding to dicounted sum games:

$$x_i = \min_{(v_i, v_j) \in E} (1 - \lambda) w(v_i, v_j) + \lambda x_j \quad \text{if } x_i \in V_1$$

$$x_i = \max_{(v_i, v_j) \in E} (1 - \lambda) w(v_i, v_j) + \lambda x_j \quad \text{if } x_i \in V_2$$

can be encoded with polynomially many bits.

**Corollary**

Discounted-sum games, mean-payoff games and parity games are in $\mathrm{UP} \cap \mathrm{coUP}$.

# One-player games

**Theorem**

One-player mean-payoff and discounted-sum games can be solved in polynomial time.

- Mean-payoff games are solved through Karp's minimum cycle mean.
- Discounted-sum games are solved through linear programming.