GandALF: weighted automata

Jan Otop

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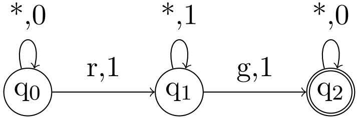
Plan for today

- Weighted automata over value functions
- Decidability of the emptiness problem
- Undecidability of the universality problem
- Mean-payoff pushdown games
- Weighted automata over semirings

Weighted automata

A weighted automaton is an automaton with transitions labeled by $\mathbb{Z}$.
Formally, $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F, C)$, where $C \mapsto \mathbb{Z}$.



A run π on a word generates a sequence of weights: $0\ 0\ 1\ 1\ 0\ \ldots$

Run's value $\mathcal{A}(\pi) = f(1\ 0\ 3\ 0\ \ldots)$ — f is a value function:

- $\mathrm{Sum}(a_1 \ldots a_k) = \sum_{i=1}^{k} a_i$, $\mathrm{Sum}^+, \mathrm{Sum}^B$, and
- $\mathrm{LimAvg}(a_1 \ldots) = \liminf_k \frac{1}{k} \sum_{i=1}^{k} a_i$.

$\mathcal{A}(w)$ is the infimum over values of runs $\mathcal{A}(\pi)$ on w.

- Finite-words and infinite-words automata.
- Various acceptance conditions.

Examples

- $\mathcal{L}_{\mathcal{A}_1}(\mathrm{w}) = |\mathrm{w}|$,
- $\mathcal{L}_{\mathcal{A}_2}(\#\mathrm{a}^{\mathrm{x}[1]}\#\mathrm{a}^{\mathrm{x}[2]}\# \ldots \mathrm{a}^{\mathrm{x}[\mathrm{n}]}) = \mathsf{min}_{\mathrm{i}>0}\, \mathrm{x}[\mathrm{i}+1] - \mathrm{x}[\mathrm{i}]$,
- $\mathcal{L}_{\mathcal{A}_3}(\mathrm{w})$ = edit distance from w to some regular language L, and
- $\mathcal{L}_{\mathcal{A}_4}(\mathrm{w})$ = average number of a's in w.

Value functions

Selection functions:

- Min, Max
- Inf, Sup, LimInf, LimSup

The sum and related functions:

- Sum
- LimAvg
- Disc

Expression power crucially depends on the value function!

No general theory of weighted automata (in this variant).

Properties

- Closed under taking minimum, multiplication and addition of a weight $(\mathrm{Sum}, \mathrm{LimAvg}, \mathrm{Disc})$.
- Not closed under maximum and additive inverse.

Decision questions

- A weighted automaton $\mathcal{A}$ defines: $\mathcal{L}_{\mathcal{A}} : \Sigma^{\omega} \to \mathbb{R} \cup \{-\infty, \infty\}$ resp. $\mathcal{L}_{\mathcal{A}} : \Sigma^{*} \to \mathbb{R} \cup \{-\infty, \infty\}$.
- Emptiness: $\exists \mathrm{w}.\mathcal{L}_{\mathcal{A}}(\mathrm{w}) \leq \lambda$?
- Universality: $\forall \mathrm{w}.\mathcal{L}_{\mathcal{A}}(\mathrm{w}) \leq \lambda$? $(\forall \mathrm{w} \exists \pi_{\mathrm{w}}.\mathrm{f}(\pi) \leq \lambda)$
- Inclusion: $\forall \mathrm{w}.\mathcal{L}_{\mathcal{A}_1}(\mathrm{w}) \leq \mathcal{L}_{\mathcal{A}_2}(\mathrm{w})$?
- Expression power and complexity of decision problems crucially depend on the value function.
- For deterministic automata, universality and emptiness have the same complexity.

Complexity of decision question for Sum and LimAvg

Theorem

The emptiness problem is decidable in polynomial time for all presented value functions.

Theorem

The universality problem is undecidable for Sum and LimAvg.

Undecidability of universality

Reduction from the haling problem for Minsky machines.

- Given a deterministic Minsky machine M with states Q.
- Alphabet $\Sigma = \mathrm{Q} \cup \{\#, 1, 2\}$.
- A configuration of M — $\#\mathrm{q}1^*2^*\#$.
- A computation of M — sequence $\mathrm{C}_1 \ldots \mathrm{C}_\mathrm{k}$:
  C_1 initial, C_k final and $(\mathrm{C}_\mathrm{i}, \mathrm{C}_{\mathrm{i}+1})$ consistent with M.
- We construct $\mathcal{A}$ such that $\mathcal{A}(\mathrm{w}) \geq 1$ iff w is a computation of M.

Complexity of decision question for Sum and LimAvg

Theorem

The emptiness problem is decidable in polynomial time for all presented value functions.

Theorem

The universality problem is undecidable for weighted automata with the Sum and LimAvg value functions.

However...

Theorem

- The universality problem is decidable for weighted automata with the Sum value function and non-negative weights.
- The inclusion problem is undecidable for weighted automata with the Sum value function and non-negative weights.

Games on automata

Consider a game of Generator and Automaton. In each turn:

- Generator produces a letter.
- Automaton picks a weighted transition.

Automaton wins iff LimAvg of weights ≤ 0.

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Simulation game: Generator plays on $\mathcal{A}_1$ and Automaton plays on $\mathcal{A}_2$.

Mean-payoff pushdown games

- $\mathcal{A}$ be a pushdown automaton with partition $\mathrm{Q} = \mathrm{Q}_1 \cup \mathrm{Q}_2$
- Game arena consists of configurations of $\mathcal{A}$.
- Player-i positions $\mathrm{Q}_\mathrm{i} \times \Gamma^*$.
- Pushdown mean-payoff games require infinite memory.

No memoryless winning strategies!

Theorem

Two-player pushdown mean-payoff games are undecidable.

Mean-payoff pushdown games — the one-player case

Theorem

One-player pushdown mean-payoff games are in P.